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Noncommutative monopoles and Riemann-Hilbert problems

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Abstract: The Bogomolny equations for Yang-Mills-Higgs monopoles follow from a system of linear equations which may be solved through a parametric Riemann-Hilbert problem. We extend this approach to noncommutative $\mathbb{R}^3$ and use it to (re)construct noncommutative Dirac, Wu-Yang, and BPS monopole configurations in a unified manner. In all cases we write down the underlying matrix-valued functions for multi-monopoles and solve the corresponding Riemann-Hilbert problems for charge one.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry, Integrable Equations in Physics.

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1. Introduction

In modern string theory, solitonic solutions to the (effective) field equations are ubiquitous. A particular class of solitons are monopoles [1], which appear as D1-branes stretched between D3-branes in type IIB superstring theory. When a constant NS B-field is turned on, gauge theory on a stack of D3-branes becomes noncommutative and supports noncommutative U(n) monopoles, including abelian ones [2]–[15].

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Irrespective of string theory, monopoles are playing an important role in the nonperturbative physics of 3+1-dimensional Yang-Mills-Higgs theory [16]–[19]. In the Prasad-Sommerfield limit interesting monopole solutions arise from a Bogomolny argument on the energy functional [20]. The simplest such configurations can be given in explicit form [21], due to the integrability of the Bogomolny equations, which in fact emerge via reduction of the self-dual Yang-Mills equations from four to three dimensions. More specifically, the Bogomolny equations may be written as the compatibility condition of an auxiliary linear system of differential equations whose solutions in turn can be obtained from a parametric Riemann-Hilbert problem. Mathematically, the latter is most naturally formulated in minitwistor space [22].

The computational task then is, firstly, to make an educated guess for a matrix-valued function f of the coordinates $(x^1, x^2, x^3) \in \mathbb{R}^3$ and of the spectral parameter $\lambda \in S^1 \subset \mathbb{C}P^1$ and, secondly, to factorize $f = \psi_+^{-1}\psi_-$, with ψ_+ and ψ_- extendable holomorphically onto complementary patches of $\mathbb{C}P^1$. This is sometimes called the "splitting method" [23]. Finally, the gauge potentials are straightforwardly recovered from the matrices $\psi_\pm$. In this fashion, the generic Bogomolny-Prasad-Sommerfield (BPS) monopoles have been constructed a long time ago, albeit only for charges $|Q| \leq 2$ explicitly [24, 25, 26, 27]. We successfully apply this technique also to the abelian Dirac as well as the point-like nonabelian Wu-Yang monopoles.

The three known methods for generating solutions to the Bogomolny equations – splitting, dressing, and Nahm's approach — all build on the fact that these equations are the compatibility conditions of some Lax pair. Fortunately, noncommutative deformation respects this integrable structure. Therefore, the just-mentioned solution-generating techniques can be generalized to the noncommutative setup. Indeed, Gross and Nekrasov [7] directly deformed the Nahm equations [28] and constructed a nonsingular abelian monopole on noncommutative $\mathbb{R}^3$. Along the same lines, noncommutative nonabelian BPS monopoles were found [9]. Their commutative limit reproduces the "ordinary" monopoles.

The above-mentioned splitting method provides an alternative approach. It has been adapted to the noncommutative situation and exploited successfully for instanton construction [29, 30]. In the present paper we apply the (deformed) splitting method to a variety of monopoles, first commutative and then noncommutative. After briefly reviewing Yang-Mills-Higgs theory and its static monopole configurations, including the noncommutative extension, in section 2, we present in section 3 the road from the Bogomolny equations to the Riemann-Hilbert problem and back to the monopole configuration in some detail. Sections 4, 5 and 6 are devoted to Dirac, Wu-Yang, and BPS monopoles, in that order, where the commutative construction precedes the noncommutative one in each case. It is remarkable that all three kinds of monopoles are parametrized by the same function appearing in their respective Riemann-Hilbert problems. Also, the derivation of noncommutative multi-monopoles is outlined, but performed only for the Wu-Yang case. For the more mathematically inclined reader, three appendices explain the twistor reformulation and its noncommutative extension, both for self-dual gauge fields and for monopoles, and demonstrate the equivalence of the general Wu-Yang and Dirac multi-monopoles.

2. Yang-Mills-Higgs model in Minkowski space

2.1 't Hooft-Polyakov monopoles

Notation. We consider the Minkowski space $\mathbb{R}^{3,1}$ with the metric $(\eta_{\mu\nu})$=diag$(-1,1,1,1)$, a real scalar (Higgs) field ϕ, a gauge potential $A{=}A_\mu dx^\mu$ and the Yang-Mills field $F{=}\mathrm{d}A{+}A{\wedge}A$ with components $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$, where $\partial_\mu:=\partial/\partial x^\mu$ and $\mu,\nu,\ldots=0,1,2,3$. The fields A and F take values in the Lie algebra $u(n)$. The matter field ϕ belongs to the adjoint representation of the group $\mathrm{U}(n)$ for any $n=1,2,\ldots$. We choose normalization of the generators T_i of this group such that $\mathrm{tr}(T_iT_j)=-\delta_{ij}$ for any representation, $i,j,\ldots=1,\ldots,n$.

Lagrangian. Consider the following Yang-Mills-Higgs lagrangian density [16, 17]:

$$\mathcal{L}=-\frac{1}{4}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}+2\eta^{\mu\nu}D_\mu\phi D_\nu\phi)-\frac{\gamma}{4}(|\phi|^2-1)^2\,, \tag{2.1}$$

where $D_\mu=\partial_\mu+[A_\mu,\ .\ ]$, $|\phi|^2:=-\,\mathrm{tr}\,\phi^2$ and $\gamma\geq 0$ is a real constant. The equations of motion following from (2.1) are

$$D_\mu F^{\mu\nu}=-\eta^{\nu\sigma}[D_\sigma\phi,\phi]\,,\qquad \eta^{\mu\nu}D_\mu D_\nu\phi=\gamma\phi(|\phi|^2{-}1)\,. \tag{2.2}$$

If the fields A and ϕ are independent on time x^0 and $A_0=0$ then the field configuration (A_a,ϕ) will be called static.

Energy. For static configurations (A_a,ϕ) the standard energy functional of the system (2.1) reduces to

$$E=-\frac{1}{4}\int \mathrm{d}^3x\big\{\mathrm{tr}(F_{ab}F_{ab}+2D_a\phi D_a\phi)+\gamma(|\phi|^2{-}1)^2\big\}\,, \tag{2.3}$$

where $a,b,\ldots=1,2,3$. This functional reaches a minimum $E=0$ when $A_a=0$, $\partial_a\phi=0$ and $|\phi|^2=1$. For $\gamma\neq 0$ nonsingular static configurations (A_a,ϕ) with finite energy exist [16, 17, 18] and are called 't Hooft-Polyakov monopoles. Finite energy ensures that static solutions to (2.2) satisfy the boundary conditions

$$\mathrm{tr}(F_{ab}F_{ab})\to 0\,,\qquad \mathrm{tr}(D_a\phi\, D_a\phi)\to 0\qquad\text{and}\qquad |\phi|^2\to 1 \tag{2.4}$$

as $r^2\equiv x^ax^a\to\infty$. From now on we restrict ourselves to the case of U(1) and SU(2) gauge groups, for the general case see e.g. [18].

Topological charges. Static SU(2) configurations (A_a,ϕ) are classified by the topological charge (number of monopoles)

$$\begin{aligned} Q&=\frac{1}{8\pi}\int \mathrm{d}^3x\ \epsilon^{abc}\,\mathrm{tr}(\partial_a\phi\partial_b\phi\partial_c\phi)=\frac{1}{8\pi}\int_{S^2_\infty}\mathrm{d}^2s_a\ \epsilon^{abc}\,\mathrm{tr}(\phi\partial_b\phi\partial_c\phi)\\ &=\frac{1}{8\pi}\int_{S^2_\infty}\mathrm{d}^2\xi\ \epsilon^{\alpha\beta}\,\mathrm{tr}\left(\phi\,\frac{\partial\phi}{\partial\xi^\alpha}\,\frac{\partial\phi}{\partial\xi^\beta}\right), \end{aligned} \tag{2.5}$$

where ξ^α are coordinates (angles) on the sphere S^2_∞ and $\epsilon^{\alpha\beta}:=\epsilon^{\alpha\beta 3}$ with $\alpha,\beta,\ldots=1,2$. Clearly, (2.5) is the winding number $\pi_2(S^2)=\mathbb{Z}$ for the map $S^2_\infty\to S^2\subset su(2)$.

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Note that the definition (2.5) is equivalent to the following:

$$Q = -\frac{1}{8\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\ \epsilon^{abc}\,\mathrm{tr}(F_{ab}D_c\phi) = -\frac{1}{4\pi}\int_{S^2_\infty} \mathrm{d}s_a\,\mathrm{tr}\left(\frac{1}{2}\epsilon_{abc}F_{bc}\phi\right) = \frac{1}{4\pi}\int_{S^2_\infty} \mathrm{d}s_a\ \tilde{B}_a\,, \tag{2.6}$$

where $\tilde{B}_a := -\,\mathrm{tr}(\frac{1}{2}\epsilon_{abc}F_{bc}\phi)$ may be identified with an abelian magnetic field as $r \to \infty$ [16, 18, 19]. In other words, far from its core a nonabelian SU(2) monopole looks like a Dirac monopole but with finite energy due to nonsingular nature around $x^a{=}0$.

In the abelian case the topological charge is defined directly via the flux. Namely, the total magnetic flux through a sphere surrounding the origin is

$$\Phi = \int_{S^2} \mathrm{d}s_a\ B_a := \int_{S^2} \mathrm{d}s_a \frac{\mathrm{i}}{2}\epsilon_{abc}F_{bc} = 4\pi Q\,, \tag{2.7}$$

where B_a is the magnetic field and Q is an integer. The Dirac monopole is defined on the space $\mathbb{R}^3\backslash\{0\}$, which is topologically equivalent to $S^2\times\mathbb{R}^+$. The flux (2.7) calculates the first Chern class of the U(1) line bundle over S^2 and can also be interpreted as the winding number $\pi_1(S^1) = \mathbb{Z}$ of the map $S^1 \to \mathrm{U}(1)$, where $S^1 \subset S^2$ is an equator in S^2 and U(1) is the gauge group. The energy of the abelian (Dirac) monopole is infinite. Any superposition of Dirac monopoles is a smooth solution on $\mathbb{R}^3\backslash\{\vec{a}_1,\ldots,\vec{a}_n\}$, where $\vec{a}_k = (a^1_k, a^2_k, a^3_k)$ for $k = 1,\ldots,n$ define the monopole locations.

2.2 Bogomolny-Prasad-Sommerfield monopoles

Static finite-energy nonabelian solutions of eqs. (2.2) exist for any $\gamma \geq 0$ [18], but for $\gamma \neq 0$ it is not easy to find a solution in closed form. The situation simplifies if one puts $\gamma = 0$ and keeps the boundary conditions (2.4). In this case one can derive the following (Bogomolny) inequality [20]:

$$\begin{aligned} E &= -\frac{1}{4}\int \mathrm{d}^3x\,\mathrm{tr}(F_{ab}F_{ab} + 2D_a\phi D_a\phi) \\ &= -\frac{1}{4}\int \mathrm{d}^3x\,\mathrm{tr}(F_{ab}\mp\epsilon_{abc}D_c\phi)(F_{ab}\mp\epsilon_{abd}D_d\phi) \mp \int \mathrm{d}^3x\,\mathrm{tr}\left(\frac{1}{2}\epsilon_{abc}F_{bc}D_c\phi\right) \\ &= -\frac{1}{4}\int \mathrm{d}^3x\,\mathrm{tr}(F_{ab}\mp\epsilon_{abc}D_c\phi)(F_{ab}\mp\epsilon_{abd}D_d\phi) \pm 4\pi Q \geq 4\pi|Q|\,, \end{aligned} \tag{2.8}$$

where $Q \in \mathbb{Z}$ is the magnetic charge (2.6). From (2.8) it follows that the absolute energy minimum $E = 4\pi Q$ for a given $Q \geq 0$ occurs when the configuration (A_a, ϕ) satisfies the first order Bogomolny equations

$$F_{ab} = \epsilon_{abc}D_c\phi\,. \tag{2.9}$$

For $Q < 0$, the right hand side of (2.9) changes its sign. The finite-action solutions of the Bogomolny equations are called BPS monopoles since the first explicit $Q{=}1$ solution was obtained by Prasad and Sommerfield [21]. Note that the Bogomolny equations (2.9) are valid also for abelian monopoles but the energy will be infinite in this case. The main topic of this paper is to demonstrate the power of the splitting method for obtaining solutions to the Bogomolny equations and their noncommutative generalization.

2.3 Bogomolny equations on noncommutative $\mathbb{R}^3$

Classical field theory on the noncommutative deformation $\mathbb{R}^3_\theta$ of $\mathbb{R}^3$ may be realized in a star-product formulation or in an operator formalism.[1] While the first approach alters the product of functions on $\mathbb{R}^3$ the second one turns these functions f into linear operators $\hat{f}$ acting on a Fock space $\mathcal{H}$. The noncommutative space $\mathbb{R}^3_\theta$ may then be defined by declaring its coordinate functions $\hat{x}^1, \hat{x}^2, \hat{x}^3$ to obey the Heisenberg algebra relations

$$[\hat{x}^a\,,\,\hat{x}^b] = \mathrm{i}\,\theta^{ab} \tag{2.10}$$

with a constant antisymmetric tensor θ^{ab}.

The coordinates can be chosen in such a way that the only nonvanishing entries of the matrix $(\theta^{\mu\nu})$ read

$$\theta^{12} = -\theta^{21} =: \theta \geq 0\,; \tag{2.11}$$

hence, the coordinate $\hat{x}^3$ is actually commutative and we shall retain the notation x^3 for it. In terms of the complex combinations (to be used later)

$$\hat{y} = \hat{x}^1 + \mathrm{i}\hat{x}^2 \qquad \text{and} \qquad \hat{\bar{y}} = \hat{x}^1 - \mathrm{i}\hat{x}^2 \tag{2.12}$$

the basic commutation relations (2.10) become

$$[\hat{y}\,,\,\hat{\bar{y}}] = 2\,\theta\,, \qquad \text{and all other commutators vanish}\,. \tag{2.13}$$

Clearly, $\hat{y}$ and $\hat{\bar{y}}$ are (up to a rescaling by $\sqrt{2\theta}$) harmonic-oscillator creation and annihilation operators, respectively. The corresponding Fock space $\mathcal{H}$ is spanned by the basis states

$$|k\rangle = (2\theta k!)^{-\frac{1}{2}}\,\hat{y}^k\,|0\rangle \qquad \text{for } k = 0, 1, 2, \dots\,, \tag{2.14}$$

so that one has the representation

$$\hat{f} = \sum_{k,\ell=0}^{\infty} f_{k\ell}(x^3)\,|k\rangle\langle\ell| \tag{2.15}$$

for any linear operator on $\mathcal{H}$. We further recall that, in the operator realization $f \mapsto \hat{f}$, derivatives of f get mapped according to

$$\partial_y f \mapsto -\theta^{-1}\,[\hat{\bar{y}}, \hat{f}] =: \hat{\partial}_y \hat{f} \qquad \text{and} \qquad \partial_{\bar{y}} f \mapsto \theta^{-1}\,[\hat{y}, \hat{f}] =: \hat{\partial}_{\bar{y}} \hat{f} \tag{2.16}$$

while $\partial_3 f \mapsto \partial_3 \hat{f}$. Finally, we have to replace

$$\int_{\mathbb{R}^3} \mathrm{d}^3x\, f \mapsto \int_{\mathbb{R}} \mathrm{d}x^3\; 2\pi\theta\, \mathrm{Tr}_{\mathcal{H}}\, \hat{f}\,. \tag{2.17}$$

With this, the noncommutative energy functional (for $\gamma{=}0$) becomes

$$E_\theta = -\frac{\pi\theta}{2}\int_{\mathbb{R}} \mathrm{d}x^3\; \mathrm{Tr}_{\mathcal{H}}\, \mathrm{tr}(\hat{F}_{ab}\hat{F}_{ab} + 2\hat{D}_a\hat{\phi}\hat{D}_a\hat{\phi})\,, \tag{2.18}$$

[1] For a review see, e.g. [31, 32].

where, of course,

$$\hat{F}_{ab} = \hat{\partial}_a \hat{A}_b - \hat{\partial}_b \hat{A}_a + [\hat{A}_a, \hat{A}_b] \qquad \text{and} \qquad \hat{D}_a \hat{\phi} = \hat{\partial}_a \hat{\phi} + [\hat{A}_a, \hat{\phi}] \,. \tag{2.19}$$

Like in the commutative case, E_θ is minimized for a given $Q \geq 0$ by the noncommutative BPS monopole configuration, which satisfies the noncommutative Bogomolny equation

$$\hat{F}_{ab} = \epsilon_{abc} \hat{D}_c \hat{\phi} \,. \tag{2.20}$$

Our main task is a systematic construction of solutions to this equation by way of a Riemann-Hilbert problem.

3. Monopoles via Riemann-Hilbert problems

3.1 Commutative case

Reduction of SDYM. The Bogomolny equations (2.9) in $\mathbb{R}^3$ can be obtained by dimensional reduction from the self-dual Yang-Mills (SDYM) equations in the euclidean space $\mathbb{R}^4$,

$$*F = F \tag{3.1}$$

where $*$ denotes the Hodge star operator. Concretely, one puts $A_4 =: \phi$ and demands that all fields be translational invariant along the x^4-axis,

$$T\,(A_a, \phi) := \partial_4 (A_a, \phi) = 0 \qquad \text{for } a = 1, 2, 3 \,. \tag{3.2}$$

This observation helps to show the integrability of the Bogomolny equations. More explicitly, powerful methods for solving the SDYM equations can be applied towards constructing monopole solutions as well. In this section, we briefly describe one of these techniques. It reduces the task to a parametric Riemann-Hilbert problem, whose solutions for various situations will be presented in subsequent sections. From a higher mathematical vantage point, this framework has a geometric description in terms of twistor and minitwistor spaces which is discussed in appendices A, B and C.

Linear system. The SDYM equations can be reformulated as the compatibility condition of two linear differential equations for an auxiliary $n \times n$ matrix function $\psi(x, \lambda)$ [33, 34]. The latter depends holomorphically on a new variable ('spectral parameter') λ which lies in the extended complex plane $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. Upon dimensional reduction from $\mathbb{R}^4$ to $\mathbb{R}^3$ of this linear system of equations one obtains a reduced linear system whose compatibility condition yields the Bogomolny equations. It is, however, advantageous not to discard the ignorable coordinate x^4 prematurely but to allow $\psi(x, \lambda)$ to still depend on it.

To become concrete, let us introduce complex coordinates

$$y = x^1 + \mathrm{i}x^2 \,, \qquad z = x^3 - \mathrm{i}x^4 \,, \qquad \bar{y} = x^1 - \mathrm{i}x^2 \,, \quad \bar{z} = x^3 + \mathrm{i}x^4 \tag{3.3}$$

and put

$$A_y = \frac{1}{2}(A_1 - \mathrm{i}A_2) \,, \qquad A_z = \frac{1}{2}(A_3 + \mathrm{i}\phi) \,, \qquad A_{\bar{y}} = \frac{1}{2}(A_1 + \mathrm{i}A_2) \,, \qquad A_{\bar{z}} = \frac{1}{2}(A_3 - \mathrm{i}\phi) \,. \tag{3.4}$$

With $D_a := \partial_a + A_a$ and $D_4 := \partial_4 + \phi$ and furthermore (A_a, ϕ) being independent of x^4, the Bogomolny equations (2.9) read

$$[D_y, D_z] = 0\,, \qquad [D_{\bar{y}}, D_{\bar{z}}] = 0\,, \qquad [D_y, D_{\bar{y}}] + [D_z, D_{\bar{z}}] = 0\,. \tag{3.5}$$

As advertized, they can be obtained as the compatibility condition of the linear system

$$(D_{\bar{y}} - \lambda D_z)\,\psi(x,\lambda) = 0 \qquad \text{and} \qquad (D_{\bar{z}} + \lambda D_y)\,\psi(x,\lambda) = 0\,, \tag{3.6}$$

where the argument x in the auxiliary $n \times n$ matrix function $\psi(x,\lambda)$ still stands for (x^1, x^2, x^3, x^4) or $(y, \bar{y}, z, \bar{z})$.

Riemann-Hilbert problems. The extended complex plane $\mathbb{C}P^1$ can be covered by two coordinate patches U_+ and U_- with

$$\mathbb{C}P^1 = U_+ \cup U_-\,, \qquad U_+ = \mathbb{C}P^1 \setminus \{\infty\}\,, \qquad U_- = \mathbb{C}P^1 \setminus \{0\}\,, \tag{3.7}$$

and coordinates λ and $\tilde{\lambda}$ on U_+ and U_-, respectively. On the intersection $U_+ \cap U_- \simeq \mathbb{C}^*$ these coordinates are related by

$$\lambda = \tilde{\lambda}^{-1}\,. \tag{3.8}$$

Suppose we are given an $n \times n$ matrix $f_{+-}(x^1, x^2, x^3, \lambda)$ which is a regular real-analytic function of

$$\lambda \in S^1 \subset U_+ \cap U_-\,, \tag{3.9}$$

i.e. it is defined on a circle for any fixed (x^1, x^2, x^3) from an open set in $\mathbb{R}^3$. Then we may consider a parametric Riemann-Hilbert problem: for each fixed (x^1, x^2, x^3) try to factorize this matrix-valued function,

$$f_{+-}(x^1, x^2, x^3, \lambda) = \psi_+^{-1}(x,\lambda)\,\psi_-(x,\lambda)\,, \tag{3.10}$$

in such a way that the two matrix factors ψ_+ and ψ_- on the right hand side are boundary values of holomorphic functions on (subsets of) U_+ and U_-, respectively. Note that $\psi_\pm$ in general depend also on x^4 while f_{+-} does not.

Suppose further that the matrix-valued function f_{+-} is chosen to depend on (x^1, x^2, x^3) only through the combination

$$\eta = y - 2\lambda x^3 - \lambda^2 \bar{y} = (1-\lambda^2)x^1 + \mathrm{i}(1+\lambda^2)x^2 - 2\lambda x^3 \qquad \text{for } \lambda \in S^1 \subset U_+ \cap U_- \tag{3.11}$$

in a holomorphic fashion, i.e. we have $f_{+-}(\eta, \lambda)$ only.[2] From $\partial_{\bar{\eta}} f_{+-} = 0 = \partial_{\bar{\lambda}} f_{+-}$ and the expression (3.11) for η it follows that

$$(\partial_{\bar{y}} - \lambda\partial_z) f_{+-} = (\partial_{\bar{z}} + \lambda\partial_y) f_{+-} = 0\,. \tag{3.12}$$

[2] Our notation is slightly sloppy: We denote both $f_{+-}(\eta,\lambda)$ and $f'_{+-}(x,\lambda) \equiv f_{+-}(\eta(x,\lambda),\lambda)$ by the same symbol.

Substituting (3.10) into (3.12) and using the (generalized) Liouville theorem, we obtain

$$\psi_+(\partial_{\bar y}-\lambda\partial_z)\psi_+^{-1} = \psi_-(\partial_{\bar y}-\lambda\partial_z)\psi_-^{-1} = \frac{1}{2}(A_1+\mathrm{i}A_2)-\frac{\lambda}{2}(A_3+\mathrm{i}\phi)\,,$$
$$\psi_+(\partial_{\bar z}+\lambda\partial_y)\psi_+^{-1} = \psi_-(\partial_{\bar z}+\lambda\partial_y)\psi_-^{-1} = \frac{1}{2}(A_3-\mathrm{i}\phi)+\frac{\lambda}{2}(A_1-\mathrm{i}A_2)\,, \qquad (3.13)$$

where the right hand sides are linear in λ and define four functions $A_a(x)$ and $\phi(x)$ independent of x^4. Hence, starting with a parametric Riemann-Hilbert problem on $S^1\subset\mathbb{C}P^1$, we arrived to the linear system (3.6) with a matrix function ψ_+ or ψ_- instead of ψ. By putting $\lambda=0$ we get

$$\frac{1}{2}(A_1+\mathrm{i}A_2)=\psi_+(\lambda)\partial_{\bar y}\psi_+^{-1}(\lambda)|_{\lambda=0}\,,\qquad \frac{1}{2}(A_3-\mathrm{i}\phi)=\psi_+(\lambda)\partial_{\bar z}\psi_+^{-1}(\lambda)|_{\lambda=0}\,. \qquad (3.14)$$

Similarly, we may take the limit $\lambda\to\infty$ to obtain

$$\frac{1}{2}(A_1-\mathrm{i}A_2)=\psi_-(\lambda)\partial_{y}\psi_-^{-1}(\lambda)|_{\lambda=\infty}\,,\qquad \frac{1}{2}(A_3+\mathrm{i}\phi)=\psi_-(\lambda)\partial_{z}\psi_-^{-1}(\lambda)|_{\lambda=\infty}\,. \qquad (3.15)$$

In this way, splitting the matrix-valued function f_{+-} automatically yields solutions (A_a,ϕ) to the Bogomolnyi equations (2.9).

Reality conditions. The components of a gauge potential and the Higgs field constructed in this way will, however, in general not be antihermitean. In order to insure that $A_a^\dagger=-A_a$ and $\phi^\dagger=-\phi$ we may impose the following 'reality' conditions on the matrices f_{+-} and $\psi_\pm$:

$$f_{+-}^\dagger(x,-\bar\lambda^{-1}) = f_{+-}(x,\lambda)\,, \qquad (3.16)$$
$$\psi_+^\dagger(x,-\bar\lambda^{-1}) = \psi_-^{-1}(x,\lambda)\,, \qquad (3.17)$$

which are compatible with the Riemann-Hilbert problem. Actually, the 'reality' condition (3.16) on f_{+-} already guarantees the possibility of (3.17). Namely, if $\psi_\pm$ do not satisfy (3.17) then one may perform a *nonunitary* gauge transformation

$$\psi_\pm\mapsto\psi_\pm^g=g^{-1}\psi_\pm \quad\text{inducing}\quad A_a\mapsto A_a^g=g^{-1}A_a\,g+g^{-1}\partial_a\,g \quad\text{and}\quad \phi\mapsto\phi^g=g^{-1}\phi g\,, \qquad (3.18)$$

where g is to be computed from

$$g^2=\psi_-(\lambda)\,\psi_+^\dagger(-\bar\lambda^{-1})=\psi_+(\lambda)\,\psi_-^\dagger(-\bar\lambda^{-1})=(g^\dagger)^2 \qquad (3.19)$$

and is independent of λ. This gauge transformation leaves f_{+-} inert but produces 'real' auxiliary functions $\psi_\pm^g$ and, hence, an antihermitean configuration (A_a^g,ϕ^g). The residual gauge freedom is of the same form but with $g'\in\mathrm{U}(n)$. To summarize, the task of constructing real monopole solutions of the Bogomolny equations can be reformulated as finding solutions to a (parametric) Riemann-Hilbert problem supplemented by the 'reality' conditions (3.16), (3.17). For more detailed discussion and references see the appendices.

3.2 Noncommutative case

In order to noncommutatively deform the above reformulation of the Bogomolny equations in terms of a Riemann-Hilbert problem, we replace the coordinates x^1 and x^2 by operators $\hat{x}^1$ and $\hat{x}^2$ or, in complex notation, y and $\bar{y}$ by $\hat{y}$ and $\hat{\bar{y}}$, respectively. In keeping with the notation for the commutative case, we introduce the combinations

$$\hat{\eta} = \hat{y} - 2\lambda x^3 - \lambda^2\hat{\bar{y}} = (1-\lambda^2)\hat{x}^1 + \mathrm{i}(1+\lambda^2)\hat{x}^2 - 2\lambda x^3 \tag{3.20}$$

and consider a matrix-valued function $\hat{f}_{+-}(\hat{\eta},\lambda)$ depending analytically on $\lambda \in S^1 = \{\lambda \in \mathbb{C}P^1 : |\lambda| = 1\}$.

Since all equations of the previous subsection are of matrix character their generalization to the noncommutative situation is formally trivial: all matrix entries are to be replaced by the corresponding operator-valued entities. In other words, we simply substitute operators $\hat{x}^a$ instead of coordinates x^a, retaining x^3 (and of course λ) as commutative. Finally, a correspondence between solutions $\big(\hat{A}_a(\hat{x}), \hat{\phi}(\hat{x})\big)$ of the noncommutative Bogomolny equations and solutions $\big(\hat{\psi}_+(\hat{x},\lambda), \hat{\psi}_-(\hat{x},\lambda)\big)$ of an operator Riemann-Hilbert problem for $\hat{f}_{+-}(\hat{\eta},\lambda)$ is described by formulae generalizing (3.13),

$$\begin{aligned}
\hat{\psi}_+(\partial_{\bar{y}} - \lambda\partial_z)\hat{\psi}_+^{-1} &= \hat{\psi}_-(\partial_{\bar{y}} - \lambda\partial_z)\hat{\psi}_-^{-1} = \frac{1}{2}(\hat{A}_1 + \mathrm{i}\hat{A}_2) - \frac{\lambda}{2}(\hat{A}_3 + \mathrm{i}\hat{\phi})\,, \\
\hat{\psi}_+(\partial_{\bar{z}} + \lambda\partial_y)\hat{\psi}_+^{-1} &= \hat{\psi}_-(\partial_{\bar{z}} + \lambda\partial_y)\hat{\psi}_-^{-1} = \frac{1}{2}(\hat{A}_3 - \mathrm{i}\hat{\phi}) + \frac{\lambda}{2}(\hat{A}_1 - \mathrm{i}\hat{A}_2)\,.
\end{aligned} \tag{3.21}$$

So, for a given splitting of an operator-valued function, $f_{+-}(\hat{\eta}(\hat{x},\lambda),\lambda) = \hat{\psi}_+^{-1}(\hat{x},\lambda)\hat{\psi}_-(\hat{x},\lambda)$, one can calculate $(\hat{A}_a, \hat{\phi})$ from

$$\begin{aligned}
\frac{1}{2}(\hat{A}_1 + \mathrm{i}\hat{A}_2) &= \hat{\psi}_+(\lambda)\partial_{\bar{y}}\hat{\psi}_+^{-1}(\lambda)|_{\lambda=0}\,, \qquad & \frac{1}{2}(\hat{A}_3 - \mathrm{i}\hat{\phi}) &= \hat{\psi}_+(\lambda)\partial_{\bar{z}}\hat{\psi}_+^{-1}(\lambda)|_{\lambda=0}\,, \\
\frac{1}{2}(\hat{A}_1 - \mathrm{i}\hat{A}_2) &= \hat{\psi}_-(\lambda)\partial_y\hat{\psi}_-^{-1}(\lambda)|_{\lambda=\infty}\,, \qquad & \frac{1}{2}(\hat{A}_3 + \mathrm{i}\hat{\phi}) &= \hat{\psi}_-(\lambda)\partial_z\hat{\psi}_-^{-1}(\lambda)|_{\lambda=\infty}\,.
\end{aligned} \tag{3.22}$$

By construction these fields satisfy the noncommutative Bogomolny equations. Fore more details see the appendices.

4. Dirac monopoles

4.1 Dirac monopoles on commutative $\mathbb{R}^3$

Here we want to rederive the famous Dirac monopole solution [35] via splitting of an appropriately chosen function f_{+-} depending holomorphically on the coordinates η and λ.

Family of curves and circles. Recall that the Dirac monopole is a solution defined not on the whole space $\mathbb{R}^3$ but on $\mathbb{R}^3\backslash\{0\}$. For a given point $(x^1, x^2, x^3) \in \mathbb{R}^3$ with $\{x^a\} \neq \{0\}$, $\eta(x,\lambda)$ vanishes in two points $\lambda_1(x)$ and $\lambda_2(x) = -1/\bar{\lambda}_1(x)$ on $\mathbb{C}P^1$. Namely, from $\eta(x,\lambda) = 0$ we have

$$y - 2\lambda x^3 - \lambda^2\bar{y} = -(\lambda - \lambda_1)(\lambda - \lambda_2)\bar{y} = 0 \qquad \text{at} \qquad \lambda = \lambda_1 \quad \text{and} \quad \lambda = \lambda_2\,, \tag{4.1}$$

where

$$\lambda_1 = -\frac{r+x^3}{\bar{y}} \qquad \text{and} \qquad \lambda_2 = \frac{r-x^3}{\bar{y}} = -\frac{1}{\bar{\lambda}_1} \qquad \text{with } r^2 := y\bar{y} + x^3x^3\,. \tag{4.2}$$

Here and in the following we suppress the implicit x dependence of $\lambda_{1,2}$ for brevity. We assume that $|\lambda_1| \leq |\lambda_2|$ and exchange their labels if this is not so. Now, consider a circle S^1 in $\mathbb{C}P^1$ defined as

$$S^1 = \begin{cases} \{\ \lambda \in \mathbb{C}P^1:\quad |\lambda| = 1 \quad \text{if} \quad |\lambda_1| < 1 < |\lambda_2|\ \} \\ \{\ \lambda \in \mathbb{C}P^1:\quad |\lambda - \varepsilon\lambda_1| = 1 \text{ with } 0{<}\varepsilon{<}1 \quad \text{if} \quad |\lambda_1| = |\lambda_2| = 1\ \} \end{cases} \tag{4.3}$$

where in the second case we have $|\lambda_1{-}\varepsilon\lambda_1| < |\lambda{-}\varepsilon\lambda_1| < |\lambda_2{-}\varepsilon\lambda_1|$ since $|\lambda_1{-}\varepsilon\lambda_1| = 1{-}\varepsilon$, $|\lambda{-}\varepsilon\lambda_1| = 1$ and $|\lambda_2{-}\varepsilon\lambda_1| = 1{+}\varepsilon$. So, the circle S^1 separates the points $\lambda{=}0$ and $\lambda{=}\lambda_1$ from the points $\lambda{=}\infty$ and $\lambda{=}\lambda_2$, i.e. the two zeros of the coordinate function $\eta(x,\lambda)$ lie in different domains.

Function to be factorized. In order to construct the Dirac monopole, we choose for our holomorphic function the simplest one possible,

$$f_{+-} = \frac{\lambda}{\eta} \qquad \Longleftrightarrow \qquad f_{-+} := (f_{+-})^{-1} = \frac{\eta}{\lambda}\,, \tag{4.4}$$

defined for $\lambda \in U_+ \cap U_-$ and $\eta \neq 0$. Taking f_{-+} to be linear in η determines the explicit λ dependence by requiring the reality condition (3.16), which in the variables η and λ takes the form

$$f^\dagger_{+-}(-\bar{\lambda}^{-2}\bar{\eta}\,, -\bar{\lambda}^{-1}) = f_{+-}(\eta,\lambda)\,. \tag{4.5}$$

Solutions of Riemann-Hilbert problems. Let us consider $f_{-+} = \eta/\lambda$ and restrict λ to a circle S^1 as defined in (4.3) on which both f_{+-} and f_{-+} are well defined. The function f_{-+} can be split on S^1 as follows:

$$f_{-+} = \frac{1}{\lambda}(y{-}2\lambda x^3{-}\lambda^2\bar{y}) = (\xi_+ + \lambda^{-1}y\xi_+^{-1})(\xi_+ - \lambda\xi_+^{-1}\bar{y}) =: (\psi_-^S(x,\lambda))^{-1}\,\psi_+^S(x,\lambda)\,, \tag{4.6}$$

where

$$\xi_+ = (r - x^3)^{1/2} = (\lambda_2\bar{y})^{1/2} = (-\lambda_1^{-1}y)^{1/2} \tag{4.7}$$

with $r^2 = x^ax^a = y\bar{y} + x^3x^3$. In fact, eqs. (4.6) give us an x^4 independent solution

$$\psi_+^S = \xi_+ - \lambda\xi_+^{-1}\bar{y} \qquad \text{and} \qquad (\psi_-^S)^{-1} = \xi_+ + \lambda^{-1}y\xi_+^{-1} \tag{4.8}$$

of the Riemann-Hilbert problem on S^1. One can write down another solution

$$f_{-+} = \frac{1}{\lambda}(y{-}2\lambda x^3{-}\lambda^2\bar{y}) = (\bar{y}\xi_- + \lambda^{-1}\xi_-^{-1})(\xi_-y - \lambda\xi_-^{-1}) =: (\psi_-^N(x,\lambda))^{-1}\,\psi_+^N(x,\lambda)\,, \tag{4.9}$$

where

$$\psi_+^N = \xi_-y - \lambda\xi_-^{-1} \qquad \text{and} \qquad (\psi_-^N)^{-1} = \bar{y}\xi_- + \lambda^{-1}\xi_-^{-1} \tag{4.10}$$

with

$$\xi_- = (r + x^3)^{-1/2} = (-\lambda_1\bar{y})^{-1/2} = (\lambda_2^{-1}y)^{-1/2}\,. \tag{4.11}$$

It is not difficult to see that both solutions (ψ_+^S, ψ_-^S) and (ψ_+^N, ψ_-^N) of the Riemann-Hilbert problem satisfy the reality condition (3.17) and therefore lead to real (antihermitean) solutions (A_a, ϕ) of the Bogomolny equations.

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Dirac monopole. Substituting $\psi_+^S(x,\lambda{=}0) = \xi_+ = (r{-}x^3)^{1/2}$ into eqs. (3.14), we obtain

$$A^S_{\bar y} \equiv \frac{1}{2}(A^S_1 + \mathrm{i}A^S_2) = \xi_+\partial_{\bar y}\xi_+^{-1} = -\frac{y}{4r(r{-}x^3)}\,, \qquad A^S_3 = 0 \qquad \text{and} \qquad \phi^S = \frac{\mathrm{i}}{2r}\,. \tag{4.12}$$

From (4.12) it follows that

$$A^S_1 = -\frac{\mathrm{i}x^2}{2r(r{-}x^3)} \quad , \qquad A^S_2 = \frac{\mathrm{i}x^1}{2r(r{-}x^3)} \qquad \text{and} \qquad A^S_3 = 0\,, \tag{4.13}$$

which coincide with the components of the gauge potential of the Dirac monopole defined on

$$\mathbb{R}^3_S := \mathbb{R}^3\backslash\{x^1 = x^2 = 0,\ x^3 \geq 0\} \tag{4.14}$$

and with the singularity along the $x^3{>}0$ axis. Analogously, substituting $\psi^N_+(x,\lambda{=}0) = \xi_- y$ into eqs. (3.14), we obtain

$$A^N_{\bar y} \equiv \frac{1}{2}(A^N_1 + \mathrm{i}A^N_2) = \xi_-\partial_{\bar y}\xi_-^{-1} = \frac{y}{4r(r{+}x^3)}\,, \qquad A^N_3 = 0 \qquad \text{and} \qquad \phi^N = \frac{\mathrm{i}}{2r} \tag{4.15}$$

and therefore

$$A^N_1 = \frac{\mathrm{i}x^2}{2r(r{+}x^3)}\,, \qquad A^N_2 = -\frac{\mathrm{i}x^1}{2r(r{+}x^3)} \qquad \text{and} \qquad A^N_3 = 0\,, \tag{4.16}$$

which coincide with the components of the Dirac monopole gauge potential well defined on

$$\mathbb{R}^3_N := \mathbb{R}^3\backslash\{x^1 = x^2 = 0,\ x^3 \leq 0\}\,, \tag{4.17}$$

i.e. everywhere on $\mathbb{R}^3\backslash\{0\}$ besides the negative x^3-axis. Note that $\mathbb{R}^3_S$ and $\mathbb{R}^3_N$ in (4.14) and (4.17) denote the southern and northern patches, respectively, of a two-sphere S^2 surrounding the origin. On the intersection of the above domains the configurations (A^N,ϕ^N) and (A^S,ϕ^S) are well defined and related by a transformation

$$A^N = A^S + \mathrm{d}\,\ln\left(\frac{\bar y}{y}\right)^{1/2} \qquad \text{and} \qquad \phi^N = \phi^S \tag{4.18}$$

which provides a global description of the Dirac monopole on $\mathbb{R}^3\backslash\{0\}$ [36].

Dirac multi-monopoles. It is obvious that for obtaining abelian multi-monopoles via solving Riemann-Hilbert problems, as a function to be factorized one may take the product

$$f^{(n)}_{-+} = \prod_{k=1}^{n}\frac{\eta_k}{\lambda} \qquad \text{where} \quad \eta_k = (1{-}\lambda^2)(x^1{-}a^1_k) + \mathrm{i}(1{-}\lambda^2)(x^2{-}a^2_k) - 2\lambda(x^3{-}a^3_k) =: \eta - h_k \tag{4.19}$$

and the $3n$ real parameters $(a^1_k, a^2_k, a^3_k) = (\vec{a}_k)$ define the positions of n monopoles in $\mathbb{R}^3$. We restrict $f^{(n)}_{-+}$ to a contour on $\mathbb{C}P^1$ which avoids all zeros $\lambda_{1,k}$ and $\lambda_{2,k}$ of η_k so that $f^{(n)}_{-+}$ is invertible on it. Using formulae (4.6)-(4.11) and abbreviating the relative coordinates $x^b - a^b_k =: x^b_k$, we easily split the factors in (4.19) individually as[3]

$$\frac{\eta_k}{\lambda} = (\psi^S_-(x_k,\lambda))^{-1}\psi^S_+(x_k,\lambda) = (\psi^N_-(x_k,\lambda))^{-1}\psi^N_+(x_k,\lambda) \tag{4.20}$$

[3]Note that in this section x_k stands for (x^a_k) and x denotes (x^a) since the splitting is independent of x^4.

and obtain

$$
\begin{aligned}
f_{-+}^{(n)} &= \left[\prod_{k=1}^{n}\psi_-^S(x_k,\lambda)\right]^{-1}\left[\prod_{\ell=1}^{n}\psi_+^S(x_\ell,\lambda)\right] =: (\psi_-^{S,n}(x,\lambda))^{-1}\psi_+^{S,n}(x,\lambda) \\
&= \left[\prod_{k=1}^{n}\psi_-^N(x_k,\lambda)\right]^{-1}\left[\prod_{\ell=1}^{n}\psi_+^N(x_\ell,\lambda)\right] =: (\psi_-^{N,n}(x,\lambda))^{-1}\psi_+^{N,n}(x,\lambda)\,. \qquad (4.21)
\end{aligned}
$$

Therefore,

$$
\psi_+^{S,n}(x,\lambda{=}0) = \prod_{k=1}^{n}\xi_+(x_k^a) =: \xi_+^{(n)} \qquad \text{and} \qquad \psi_+^{N,n}(x,\lambda{=}0) = \prod_{k=1}^{n}\xi_-(x_k^a)y_k =: \xi_-^{(n)}y \qquad (4.22)
$$

and analogously for $\psi_-^{S,n}$ and $\psi_-^{N,n}$.

Substituting (4.22) into (3.14), we obtain

$$
\begin{aligned}
A_{\bar y}^{S,n} &= \xi_+^{(n)}\partial_{\bar y}(\xi_+^{(n)})^{-1} = \sum_{k=1}^{n}\xi_+(x_k^a)\partial_{\bar y}\xi_+^{-1}(x_k^a) =: \sum_{k=1}^{n}A_{\bar y}^S(x_k^a)\,, \\
A_3^{S,n} &= 0 \qquad \text{and} \qquad \phi^{S,n} = \sum_{k=1}^{n}\frac{\mathrm{i}}{2r_k} \qquad \text{with} \quad r_k^2 = \sum_{a=1}^{3}x_k^a x_k^a\,, \qquad (4.23)
\end{aligned}
$$

and analogously for $A_{\bar y}^{N,n}$ with $A_3^{N,n} = 0$ and $\phi^{N,n} = \phi^{S,n}$. Thus, solving Riemann-Hilbert problems on an appropriate circle surrounding all $\lambda{=}\lambda_{1,k}$, we obtain a linear superposition of n Dirac monopole solutions regular on the space $\mathbb{R}^3\backslash\{\vec a_1,\dots,\vec a_n\}$.

4.2 Noncommutative Dirac monopoles

Abelian solutions of the noncommutative Bogomolny equations have been constructed in closed form by Gross and Nekrasov [7] in the noncomutative generalization of the Nahm approach [28]. Here we show how their solution (the noncommutative Dirac monopole) can be obtained by solving the appropriate Riemann-Hilbert problem. Note that the string-like singularity of the commutative Dirac monopole is absent in the noncommutative configuration [7]. For this reason we can and will restrict ourselves to a noncommutative generalization of the southern patch quantities $\psi_\pm^S$, ξ_+ etc. derived in section 4.1 and in the following omit the 'S' superscript indicating the singularity of the standard Dirac monopole on the positive x^3-axis.

Proper circles. In order to pass to the noncommutative Riemann-Hilbert problem we promote the coordinates $(y,\bar y)$ to operators $(\hat y,\hat{\bar y})$ acting in a one-oscillator Fock space $\mathcal{H}$ from the left or in its dual $\mathcal{H}^*$ from the right. To mimic the commutative construction we should first of all describe circles $S^1\subset\mathbb{C}P^1$ such that the operator

$$
\hat\eta = \hat y - 2\lambda x^3 - \lambda^2\hat{\bar y} \qquad (4.24)
$$

does not vanish on vectors from $\mathcal{H}$ or $\mathcal{H}^*$ if $\lambda\in S^1$. Surprisingly, in the noncommutative case the situation is simpler than the one discussed in section 4.1. Namely, recall that $\hat y$

is proportional to the annihilation operator a and that $\hat{\bar{y}}$ is proportional to the creation operator $a^\dagger$. Obviously, the operator $\hat{\eta}$ is a linear combination of a, $a^\dagger$, and $\mathbf{1}$. Therefore, its eigenvectors $|\varphi\rangle \in \mathcal{H}$,

$$\hat{\eta}|\varphi\rangle = \varphi|\varphi\rangle \qquad \text{with} \qquad \varphi \in \mathbb{C}, \tag{4.25}$$

are so-called squeezed (or generalized coherent) states [37, 38]. It is well known that its zero eigenvalues, $\varphi = 0$, belong to normalizable states (and therefore occur in $\mathcal{H}$) if and only if $|\lambda| < 1$ [37, 38]. Analogously, the kernel of $\hat{\eta}$ in $\mathcal{H}^*$, $\langle\varphi'|\hat{\eta} = 0$, is non-empty iff $|\lambda| > 1$. Therefore, on the circle

$$S^1 = \{\lambda \in \mathbb{C}P^1 : |\lambda| = 1\} \tag{4.26}$$

the operator $\hat{\eta}$ has no zero modes either on $\mathcal{H}$ or on $\mathcal{H}^*$. So, for $|\lambda| = 1$ the operator $\hat{\eta}$ is invertible on states with finite norm, i.e. on $\mathcal{H}$ as well as on $\mathcal{H}^*$.

Noncommutative Dirac monopole. We adhere to the simplest ansatz and assume that the holomorphic function (4.4) describing the commutative Dirac monopole keeps its form in the noncommutative case, i.e.

$$\hat{f}_{+-} = \lambda\,\hat{\eta}^{-1} \qquad \text{and} \qquad \hat{f}_{-+} = \lambda^{-1}\,\hat{\eta}\,. \tag{4.27}$$

In (4.27) we assume $|\lambda| = 1$ so that $\hat{\eta}$ is invertible. Again, the operator (4.27) is real in the sense that

$$\hat{f}^\dagger_{+-}\left(-\frac{\hat{\eta}^\dagger}{\bar{\lambda}^2}, -\frac{1}{\bar{\lambda}}\right) = \hat{f}_{+-}(\hat{\eta}, \lambda) \tag{4.28}$$

(cf. eq. (4.5)).

The operator-valued function

$$\hat{f}_{-+} = \frac{\hat{\eta}}{\lambda} = \frac{1}{\lambda}(\hat{y} - 2\lambda x^3 - \lambda^2\hat{\bar{y}}) = \sqrt{2\theta}\left(\lambda^{-1}a - \frac{2x^3}{\sqrt{2\theta}} - \lambda a^\dagger\right) \tag{4.29}$$

can be split as follows:[4]

$$\hat{f}_{-+} = (\xi + \lambda^{-1}\hat{y}\xi^{-1})\,(\xi - \lambda\xi^{-1}\hat{\bar{y}}) =: \hat{\psi}_-^{-1}(\hat{x},\lambda)\,\hat{\psi}_+(\hat{x},\lambda)\,, \tag{4.30}$$

where the operator ξ is implicitly defined by the equation

$$\xi^2 - \hat{y}\xi^{-2}\hat{\bar{y}} = -2x^3\,. \tag{4.31}$$

This equation has been solved earlier in [7]. Namely,

$$\xi(x^3) = (2\theta)^{1/4}\sum_{\ell\geq 0}\xi_\ell(x^3)\,|\ell\rangle\langle\ell| \qquad \text{with} \qquad \xi_\ell(x^3) = \sqrt{\frac{\Upsilon_{\ell-1}(\frac{x^3}{\sqrt{2\theta}})}{\Upsilon_\ell(\frac{x^3}{\sqrt{2\theta}})}}\,, \tag{4.32}$$

[4]We are grateful to N. Nekrasov for proposing this splitting.

where the special functions $\Upsilon_\ell(x)$ are given by formulae

$$\Upsilon_{-1} = 1 \qquad \text{and} \qquad \Upsilon_\ell(x) = \int_0^\infty \mathrm{d}p \frac{p^\ell}{\ell!} \exp\left(2xp - \frac{1}{2}p^2\right) \qquad \text{for } \ell \geq 0\,. \tag{4.33}$$

One can show that $\Upsilon_\ell(x)$ satisfy the recurrence relations

$$(\ell{+}1)\Upsilon_{\ell+1} = 2x\Upsilon_\ell + \Upsilon_{\ell-1} \tag{4.34}$$

and also

$$\partial_x \Upsilon_\ell(x) = 2(\ell{+}1)\Upsilon_{\ell+1}(x)\,. \tag{4.35}$$

One easily sees that (4.32) solves (4.31) due to eqs. (4.34) [7].

From (4.30) we obtain operators

$$\hat{\psi}_+(\hat{x},\lambda) = \xi - \lambda\xi^{-1}\hat{y} \qquad \text{and} \qquad \hat{\psi}_-^{-1}(\hat{x},\lambda) = \xi + \lambda^{-1}\hat{y}\xi^{-1} \tag{4.36}$$

satisfying the reality condition (3.17). Substituting (4.36) into formulae (3.22), we obtain a solution $(\hat{A}_a, \hat{\phi})$ of the noncommutative Bogomolny equations:

$$\hat{A}_1 = -\frac{1}{\theta}\,[\hat{x}_1,\xi]\,\xi^{-1}\,, \qquad \hat{A}_2 = \frac{1}{\theta}\,[\hat{x}_2,\xi]\,\xi^{-1}\,, \qquad \hat{A}_3 = 0\,, \tag{4.37}$$

$$\hat{\phi} = -\mathrm{i}(\partial_3\xi)\,\xi^{-1} = -\mathrm{i}\sum_{\ell\geq 0}\partial_3(\ln\xi_\ell(x^3))\,|\ell\rangle\langle\ell|\,. \tag{4.38}$$

This describes the noncommutative Dirac monopole. For a discussion of its properties and its D-brane interpretation see [7, 39].

Noncommutative Dirac multi-monopoles. In order to obtain noncommutative multi-monopoles one may consider the Riemann-Hilbert problem for the operator-valued function

$$\hat{f}_{-+}^{(n)} = \prod_{k=1}^{n} \frac{1}{\lambda}\,(\hat{\eta} - h_k)\,, \tag{4.39}$$

where h_k are polynomials in λ not depending on the coordinates x and satisfying

$$\overline{h_k\left(-\frac{1}{\bar{\lambda}}\right)} = -\lambda^{-2}\,h_k(\lambda) \tag{4.40}$$

with the bar denoting complex conjugation. The simplest example of such h_k is written down in eq. (4.19).

Note that it is not easy to split the operator-valued function (4.39) even in the simplest $n{=}2$ case. We will not try to do this here. Instead, in the next section we describe noncommutative generalizations of SU(2) Wu-Yang multi-monopole solutions which asymptotically (for large separation) approach to Dirac multi-monopole solutions with the U(1) group embedded into the gauge group SU(2). The exact equivalence of the Dirac multi-monopoles to the general SU(2) Wu-Yang multi-monopoles generated by the $\mathcal{A}_n$ Atiyah-Ward ansatz is proven in appendix C.

5. Wu-Yang monopoles

5.1 The Wu-Yang SU(2) monopole

A spherically-symmetric singular monopole solution of the SU(2) gauge field equations has been obtained by Wu and Yang in 1969 [40]. It was later interpreted (see e.g. [41] and references therein) as a static solution,

$$A_a = \epsilon_{abc} \frac{\sigma_c}{2\mathrm{i}} \frac{x^b}{r^2}\,, \qquad \phi = \frac{\sigma_a}{2\mathrm{i}} \frac{x^a}{r^2}\,, \tag{5.1}$$

of the Yang-Mills-Higgs equations (2.2). It is not difficult to see that this configuration satisfies also the Bogomolny equations (2.9) but has infinite energy. Initially it was thought that (5.1) is genuinely nonabelian, yet after 't Hooft's analysis [16] it was realized that it is nothing but the (abelian) Dirac monopole in disguise (see e.g. [42, 36, 43, 44]). Furthermore, the gauge potential of the finite-energy spherically symmetric BPS SU(2) monopole approaches the gauge potential in (5.1) for large r.

The equivalence of the Wu-Yang SU(2) monopole and the Dirac U(1) monopole (with one gauge group embedded into the other) can be seen as follows. Recall that the Dirac monopole is described (see section 4) by the gauge potentials A^S and A^N defined on $\mathbb{R}^3_S$ and $\mathbb{R}^3_N$, respectively. Here, $\mathbb{R}^3_S \cup \mathbb{R}^3_N = \mathbb{R}^3 \backslash \{0\}$, and on the overlap region $\mathbb{R}^3_S \cap \mathbb{R}^3_N$ the gauge potentials are related via a transition function (cf. (4.18))

$$f^{(D)}_{NS} = \left(\frac{y}{\bar{y}}\right)^{1/2} =: \mathrm{e}^{i\varphi} \tag{5.2}$$

as

$$A^N = f^{(D)}_{NS} A^S \left(f^{(D)}_{NS}\right)^{-1} + f^{(D)}_{NS} \mathrm{d}\left(f^{(D)}_{NS}\right)^{-1}. \tag{5.3}$$

Let us multiply this equation by the matrix $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and rewrite it as

$$A^N \sigma_3 = f_{NS}\; A^S \sigma_3\; (f_{NS})^{-1} \;+\; f_{NS}\; \mathrm{d}(f_{NS})^{-1}\,, \tag{5.4}$$

where

$$f_{NS} := \begin{pmatrix} f^{(D)}_{NS} & 0 \\ 0 & (f^{(D)}_{NS})^{-1} \end{pmatrix} = \begin{pmatrix} (\frac{y}{\bar{y}})^{1/2} & 0 \\ 0 & (\frac{y}{\bar{y}})^{-1/2} \end{pmatrix}. \tag{5.5}$$

Obviously, (5.4) is equivalent to (4.18).

Note now that one may split the transition matrix (5.5) as

$$f_{NS} = g_N^{-1}\, g_S\,, \tag{5.6}$$

where the 2×2 unitary matrices

$$g_N := \frac{1}{\sqrt{2r(r{+}x^3)}} \begin{pmatrix} \bar{y} & r{+}x^3 \\ -(r{+}x^3) & y \end{pmatrix} \qquad \text{and} \qquad g_S := \frac{1}{\sqrt{2r(r{-}x^3)}} \begin{pmatrix} r{-}x^3 & \bar{y} \\ -y & r{-}x^3 \end{pmatrix} \tag{5.7}$$

are well defined on $\mathbb{R}^3_N$ and $\mathbb{R}^3_S$, respectively. Substituting (5.6) into (5.4), we obtain

$$g_N\, A^N \sigma_3\, g_N^{-1} + g_N\, \mathrm{d}g_N^{-1} = g_S\, A^S \sigma_3\, g_S^{-1} + g_S\, \mathrm{d}g_S^{-1} =: A^{\mathrm{SU}(2)}\,, \tag{5.8}$$

where by construction $A^{\mathrm{SU}(2)}$ is well defined on $\mathbb{R}^3_S \cup \mathbb{R}^3_N = \mathbb{R}^3 \backslash \{0\}$. Geometrically, the existence of the splitting (5.6) means that the Dirac monopole's nontrivial U(1) bundle trivializes when being imbedded into an SU(2) bundle. The matrices (5.7) define this trivialization since $f_{NS} \mapsto f^{\text{new}}_{NS} = g_N\, f_{NS} g_S^{-1} = \mathbf{1}_2$ [36].

Substituting (5.7) and the Dirac monopole configuration (4.13) and (4.16) into (5.8), we obtain $A^{\mathrm{SU}(2)} = A_{\bar{y}} \mathrm{d}\bar{y} + A_y \mathrm{d}y + A_3 \mathrm{d}x^3$ with

$$A_{\bar{y}} = \frac{1}{4r^2}\begin{pmatrix} -y & 2x^3 \\ 0 & y \end{pmatrix}, \qquad A_y = \frac{1}{4r^2}\begin{pmatrix} -\bar{y} & 0 \\ 2x^3 & \bar{y} \end{pmatrix} \qquad \text{and} \qquad A_3 = \frac{1}{2r^2}\begin{pmatrix} 0 & -\bar{y} \\ y & 0 \end{pmatrix} \tag{5.9}$$

which in real coordinates coincides with (A_a) given in (5.1). Analogously, for the Higgs field we have

$$\phi^N \sigma_3 = g_N^{-1} g_S\, \phi^S \sigma_3\, g_S^{-1} g_N \quad \Longrightarrow \quad \phi := g_N\, \phi^N \sigma_3\, g_N^{-1} = g_S\, \phi^S \sigma_3\, g_S^{-1} = \frac{x^a}{r^2}\frac{\sigma_a}{2\mathrm{i}}\,, \tag{5.10}$$

which is identical to ϕ in (5.1).

The equivalence of the Wu-Yang SU(2) monopole to the Dirac abelian monopole extends to the multi-monopole situation. One may take, for instance, the abelian multi-monopole solution (4.23) and transform it into a Wu-Yang multi-monopole solution. Explicit matrices g_N and g_S may be found in [42, 43] for the case of monopoles situated along a straight line. Instead of directly generalizing the formulae presented above, we shall alternatively derive special Wu-Yang multi-monopole configurations via solving a Riemann-Hilbert problem in the following subsection. A discussion of the general Wu-Yang SU(2) multi-monopole and its equivalence to the Dirac U(1) multi-monopole is relegated to appendix C.

5.2 Wu-Yang SU(2) multi-monopoles

Matrix-valued holomorphic function. We would like to formulate a matrix Riemann-Hilbert problem appropriate for the Wu-Yang monopole. Here, the main skill consists in guessing a suitable holomorphic matrix-valued function $f^{\mathrm{SU}(2)}_{+-}$. We make the Atiyah-Ward $\mathcal{A}_1$ ansatz [5] [45, 23]

$$f^{\mathrm{SU}(2)}_{+-} = \begin{pmatrix} \rho & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} \tag{5.11}$$

containing a function $\rho = \rho(\eta, \lambda)$ holomorphic in the (local) coordinates η and λ. We restrict $f^{\mathrm{SU}(2)}_{+-}$ to our circle $S^1 \subset \mathbb{C}P^1$; then ρ depends parametrically on $(x^1, x^2, x^3) \in U \subset \mathbb{R}^3$. Furthermore, we impose on ρ the reality condition

$$\rho^\dagger(x, -\bar{\lambda}^{-1}) = \rho(x, \lambda) \tag{5.12}$$

which guarantees that $f^{\mathrm{SU}(2)}_{+-}$ satisfies the reality condition (3.16) and therefore will produce real solutions of the Bogomolny equations.

[5] For a discussion of the more general $\mathcal{A}_n$ Atiyah-Ward ansatz and the corresponding Wu-Yang multi-monopole see appendix C.

It is useful to expand ρ in a Laurent series in λ,

$$\rho = \sum_{m=-\infty}^{\infty} \rho_m \, \lambda^m = \rho_- + \rho_0 + \rho_+ \,, \tag{5.13}$$

where

$$\rho_+ := \sum_{m>0} \rho_m \, \lambda^m \qquad \text{and} \qquad \rho_- := \sum_{m<0} \rho_m \, \lambda^m \,. \tag{5.14}$$

The Laurent coefficients $\rho_m = \rho_m(x^1, x^2, x^3)$ are not functionally independent. From the restricted x-dependence of ρ (its holomorphicity in η) it follows that (c.f. (3.12))

$$(2\partial_{\bar{y}} - \lambda\partial_3)\rho = (\partial_3 + 2\lambda\partial_y)\rho = 0 \tag{5.15}$$

which implies the recursion relations

$$2\partial_{\bar{y}}\rho_{m+1} = \partial_3\rho_m = -2\partial_y\rho_{m-1} \,. \tag{5.16}$$

Riemann-Hilbert problem and solutions. It is not hard to see that $f_{+-}^{\mathrm{SU}(2)}$ can be split as

$$f_{+-}^{\mathrm{SU}(2)} = \psi_+^{-1}\,\psi_- \,, \tag{5.17}$$

where (see also [29])

$$\psi_-^{-1} = \begin{pmatrix} 1 & -\lambda^{-1} \\ -\lambda\rho_- & \rho_0 + \rho_- \end{pmatrix} \frac{1}{\sqrt{\rho_0}} \,, \qquad \psi_- = \frac{1}{\sqrt{\rho_0}} \begin{pmatrix} \rho_0 + \rho_- & \lambda^{-1} \\ \lambda\rho_- & 1 \end{pmatrix} , \tag{5.18}$$

$$\psi_+ = \frac{1}{\sqrt{\rho_0}} \begin{pmatrix} 1 & \lambda^{-1}\rho_+ \\ \lambda & \rho_0 + \rho_+ \end{pmatrix} , \qquad \psi_+^{-1} = \begin{pmatrix} \rho_0 + \rho_+ & -\lambda^{-1}\rho_+ \\ -\lambda & 1 \end{pmatrix} \frac{1}{\sqrt{\rho_0}} \,. \tag{5.19}$$

Formulae (5.17) – (5.19) solve our parametric Riemann-Hilbert problem on the circle S^1 in an x^4-independent manner. One can show that the above matrices $\psi_\pm(x,\lambda)$ satisfy the reality condition (3.17). Substituting (5.19) into formulae (3.14) and using the recursion relations (5.16), we get

$$A_a = -\epsilon_{abc}\,\frac{\sigma_c}{2\mathrm{i}}\,\rho_0^{-1}\,\partial_b\rho_0 \qquad \text{and} \qquad \phi = -\frac{\sigma_a}{2\mathrm{i}}\,\rho_0^{-1}\,\partial_a\rho_0 \,, \tag{5.20}$$

where, due to (5.16), the function ρ_0 is a solution of the Laplace equation

$$(\partial_1^2 + \partial_2^2 + \partial_3^2)\rho_0 = 0 \tag{5.21}$$

on an open subset U of $\mathbb{R}^3$.

Charge one monopole. On $U = \mathbb{R}^3 \backslash \{0\}$, the only spherically symmetric solution (up to an additive constant and irrelevant normalization) is

$$\rho_0 = \frac{1}{2r} \,. \tag{5.22}$$

Substitution it into (5.20) immediately yields the Wu-Yang monopole solution (5.1).

It is noteworthy that one can find the explicit form not only of ρ_0 but also of the complete holomorphic function ρ. Therefore, we can give a closed expression for the matrix-valued function $f_{+-}^{\mathrm{SU}(2)}$ in (5.11) which describes the Wu-Yang monopole. In order to reconstruct the function ρ from ρ_0, we recall the reality condition (5.12) as

$$\rho^\dagger(-\bar{\lambda}^{-2}\bar{\eta}\,,-\bar{\lambda}^{-1}) = \rho(\eta,\lambda)\,. \tag{5.23}$$

Since the single-monopole solution (5.1) contains no free (dimensionful) parameters such as a scale or a position, this condition implies that ρ is a function of the combination $\frac{\eta}{\lambda} = \lambda^{-1}y - 2x^3 - \lambda\bar{y}$ only. The length dimension of ρ_0 in (5.22) then determines ρ to be proportional to $(\eta/\lambda)^{-1}$.

For explicit verification we fix the point $x \in \mathbb{R}^3 \setminus \{0\}$ which yields $\lambda_{1,2}$ according to (4.2). Assuming as in section 4.1 that $|\lambda_1| < |\lambda| < |\lambda_2|$, we obtain

$$\begin{aligned} \frac{\lambda}{\eta} &= \frac{\lambda}{(\lambda-\lambda_1)(\lambda_2-\lambda)\,\bar{y}} = \frac{1}{(\lambda_2-\lambda_1)\,\bar{y}}\left(\frac{\lambda_1}{\lambda-\lambda_1} + \frac{\lambda_2}{\lambda_2-\lambda}\right) \\ &= \frac{1}{(\lambda_2-\lambda_1)\,\bar{y}}\left(\frac{\lambda_1/\lambda}{1-\lambda_1/\lambda} + \frac{1}{1-\lambda/\lambda_2}\right) = \frac{1}{(\lambda_2-\lambda_1)\bar{y}}\left\{1 + \sum_{\ell\geq 1}\Big(\frac{\lambda_1}{\lambda}\Big)^{\ell} + \sum_{\ell\geq 1}\Big(\frac{\lambda}{\lambda_2}\Big)^{\ell}\right\} \end{aligned} \tag{5.24}$$

and therefore indeed

$$\left(\frac{\lambda}{\eta}\right)_0 = \frac{1}{(\lambda_2-\lambda_1)\,\bar{y}} = \frac{1}{2r} = \rho_0\,. \tag{5.25}$$

Alternatively,

$$\rho_0 = \oint\limits_{|\lambda|=1} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda}\,\rho = \mathrm{res}_{\lambda=\lambda_1}\eta^{-1} = \frac{1}{(\lambda_2-\lambda_1)\,\bar{y}} = \frac{1}{2r}\,. \tag{5.26}$$

Analogously for the other case in (4.3), $|\lambda_1| = |\lambda_2| = 1$, one arrives at the equivalent solution $\rho_0 = (2(1+\varepsilon)r)^{-1}$ with $0 < \varepsilon < 1$. It is amusing that only the power -1 of $\frac{\eta}{\lambda}$ yields a spherically symmetric function ρ_0. We conclude that the Wu-Yang monopole is fully described by the function

$$\rho(\eta,\lambda) = \frac{\lambda}{\eta} = (\lambda^{-1}y - 2x^3 - \lambda\,\bar{y})^{-1}\,. \tag{5.27}$$

The equivalence of the Wu-Yang solution to the Dirac monopole solution follows also from the fact that this function coincides with the function (4.4) generating the Dirac monopole.

Multi-monopole. To generate a multi-monopole solution describing n monopoles sitting at points $(a_k^1, a_k^2, a_k^3) = (\vec{a}_k)$ with $k = 1,\dots,n$, one may take the function

$$\rho^{(n)} = \sum_{k=1}^{n}\frac{\lambda}{\eta_k} \qquad \text{with} \quad \eta_k = (1-\lambda^2)(x^1-a_k^1) + \mathrm{i}(1-\lambda^2)(x^2-a_k^2) - 2\lambda(x^3-a_k^3) =: \eta - h_k \tag{5.28}$$

restricted to a circle in $\mathbb{C}P^1$ (which does not pass through the zeros of any η_k) and insert it into the matrix (5.11). Then solving the Riemann-Hilbert problem as in (5.17)–(5.19)

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we produce the field configuration (5.20) with

$$\rho_0^{(n)} = \sum_{k=1}^{n} \frac{1}{2r_k} = \frac{1}{2}\sum_{k=1}^{n} \left[(x^1-a_k^1)^2 + (x^2-a_k^2)^2 + (x^3-a_k^3)^2\right]^{-1/2} , \tag{5.29}$$

defined on $\mathbb{R}^3\backslash\{\vec{a}_1,\ldots,\vec{a}_n\}$. It describes a Wu-Yang multi-monopole but not the most general one. This configuration asymptotically approaches to the configuration (4.23) describing n Dirac monopoles. One can see, for instance, that for $x^c \to a_k^c$ we have

$$A_a \quad\to\quad A_a^{(k)} = \varepsilon_{abc}\frac{\sigma_c}{2\mathrm{i}}\frac{x_k^b}{r_k^2} \qquad \text{and} \qquad \phi \quad\to\quad \phi^{(k)} = \frac{\sigma_a}{2\mathrm{i}}\frac{x_k^a}{r_k^2} \tag{5.30}$$

which can be transformed to the kth Dirac monopole with the help of matrices (5.7) where x_k^a and r_k are substituted for x^a and r. Then, considering a small sphere S_k^2 surrounding the point $x^c = a_k^c$, we easily find via (2.7) that the flux through any such S_k^2 produces one unit of topological charge and their summation yields n for the charge of the whole configuration.

5.3 Noncommutative Wu-Yang monopoles

Generic form of the solution. Considering the noncommutative space $\mathbb{R}^3_\theta$, we assume that the 2×2 matrix encoding noncommutative Wu-Yang U(2) monopoles[6] has the form (5.11) but with the holomorphic function ρ promoted to an operator $\hat\rho$ acting in the one-oscillator Fock space $\mathcal{H}$. We find that formulae (5.18) and (5.19) with $x^a \to \hat{x}^a$ solve the operator Riemann-Hilbert problem

$$\hat{f}_{+-}^{\mathrm{SU}(2)} = \begin{pmatrix} \hat\rho & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} = \hat\psi_+^{-1}(\hat{x},\lambda)\,\hat\psi_-(\hat{x},\lambda) \tag{5.31}$$

and eqs. (5.15)–(5.16) also hold their form for $\hat\rho = \hat\rho_+ + \hat\rho_0 + \hat\rho_-$. Substituting (5.31), (5.18) and (5.19) into formulae (3.22) and using the recursion relations (5.16), we obtain

$$\begin{aligned} \hat{A}_a &= \varepsilon_{abc}\frac{\sigma_c}{2\mathrm{i}}\left(\hat\rho_0^{\frac12}\partial_b\hat\rho_0^{-\frac12} - \hat\rho_0^{-\frac12}\partial_b\hat\rho_0^{\frac12}\right) + \frac{\mathbf{1}_2}{2}\left(\hat\rho_0^{-\frac12}\partial_b\hat\rho_0^{\frac12} + \hat\rho_0^{\frac12}\partial_b\hat\rho_0^{-\frac12}\right), \\ \hat\phi &= \frac{\sigma_a}{2\mathrm{i}}\left(\hat\rho_0^{\frac12}\partial_a\hat\rho_0^{-\frac12} - \hat\rho_0^{-\frac12}\partial_a\hat\rho_0^{\frac12}\right). \end{aligned} \tag{5.32}$$

Here, the operator $\hat\rho_0$ can be obtained from a given $\hat\rho$ via the Cauchy formula,

$$\hat\rho_0 = \oint_{|\lambda|=1} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda}\,\hat\rho\,. \tag{5.33}$$

Recall that for the invertibility of the operator $\hat\eta$ from (4.24) we assume that $|\lambda| = 1$ in (5.31) and for $\hat\rho$ (see section 4.2 for discussion of this).

[6]It is well known that in the noncommutative setup one should consider the group U(2) instead of SU(2) since the unit determinant condition is not preserved by noncommutative gauge transformations (see e.g. [46]).

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Noncommutative monopole solution. For the description of a noncommutative Wu-Yang monopole with moduli $\vec{a}_k = (a_k^1, a_k^2, a_k^3)$ we take in (5.31)

$$\hat{\rho}_{(k)} = \lambda\,\hat{\eta}_k^{-1} = \lambda\,(\hat{y}_k - 2\lambda x_k^3 - \lambda^2\hat{\bar{y}}_k)^{-1}\,, \tag{5.34}$$

where, as before,

$$\hat{y}_k = \hat{y} - (a_k^1 + \mathrm{i}a_k^2)\,, \qquad \hat{\bar{y}}_k = \hat{\bar{y}} - (a_k^1 - \mathrm{i}a_k^2) \qquad \text{and} \qquad x_k^3 = x^3 - a_k^3\,. \tag{5.35}$$

As usual (see e.g. [47] and references therein), we introduce a new basis $\{|\ell\rangle_k,\ \ell = 0,1,2,\dots\}$ in $\mathcal{H}$ via

$$\hat{y}_k|0\rangle_k = 0\,, \qquad |\ell\rangle_k = \frac{\hat{\bar{y}}_k^\ell}{\sqrt{\ell!(2\theta)^\ell}}|0\rangle_k\,, \qquad \ell = 0,1,2,\dots \tag{5.36}$$

Then consider an operator (cf. (4.32))

$$\xi_{(k)} := (2\theta)^{1/4}\sum_{\ell\geq 0}\xi_\ell(x_k^3)|\ell\rangle_k\langle\ell|_k \qquad \text{with} \qquad \xi_\ell(x_k^3) = \sqrt{\frac{\Upsilon_{\ell-1}(x_k^3/\sqrt{2\theta})}{\Upsilon_\ell(x_k^3/\sqrt{2\theta})}}\,, \tag{5.37}$$

which solves the equation

$$\xi_{(k)}^2 - \hat{y}_k\,\xi_{(k)}^{-2}\,\hat{\bar{y}}_k = -2x_k^3\,. \tag{5.38}$$

Using the operator (5.37) and formula (4.30), we obtain

$$\begin{aligned}
\hat{\rho}_{(k)0} &= \oint\limits_{|\lambda|=1}\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda}\,\hat{\rho}_{(k)} = \oint\limits_{|\lambda|=1}\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\,\hat{\eta}_k^{-1} = \oint\limits_{|\lambda|=1}\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\,(\xi_{(k)}^2 - \lambda\hat{\bar{y}}_k)^{-1}(\lambda + \hat{y}_k\xi_{(k)}^{-2})^{-1} \\
&= \oint\limits_{|\lambda|=1}\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\,(2R_k)^{-1}\left[1 - (\lambda + \hat{y}_k\xi_{(k)}^{-2})\,\hat{\bar{y}}_k\,(2R_k)^{-1}\right]^{-1}(\lambda + \hat{y}_k\xi_{(k)}^{-2})^{-1} \\
&= (2R_k)^{-1}\oint\limits_{|\lambda|=1}\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\,\left\{(\lambda + \hat{y}_k\xi_{(k)}^{-2})^{-1} + (\lambda + \hat{y}_k\xi_{(k)}^{-2})\,\hat{\bar{y}}_k\,(2R_k)^{-1}(\lambda + \hat{y}_k\xi_{(k)}^{-2})^{-1} + \cdots\right\} \\
&= (2R_k)^{-1}\,,
\end{aligned} \tag{5.39}$$

where

$$2R_k := \xi_{(k)}^2 + \hat{y}_k\,\xi_{(k)}^{-2}\,\hat{\bar{y}}_k = 2\,(\xi_{(k)}^2 + x_k^3) \tag{5.40}$$

is an invertible operator as easily seen from (5.37). Note that in the commutative limit $R_k \to r_k = \sqrt{(x_k^1)^2 + (x_k^2)^2 + (x_k^3)^2}$ in conformity with (5.25). Substituting (5.39) into (5.32), we obtain a noncommutative generalization of the Wu-Yang monopole.

Noncommutative multi-monopoles. For obtaining noncommutative Wu-Yang multi-monopole solutions one may take

$$\hat{\rho}^{(n)} = \sum_{k=1}^{n}\hat{\rho}_{(k)} \tag{5.41}$$

with $\hat{\rho}_{(k)}$ from (5.34) invertible for $|\lambda|=1$ and substitute it into (5.31) to produce the multi-monopole matrix. This is nothing but the Atiyah-Ward $\mathcal{A}_1$ ansatz reduced to three dimensions. Then, (5.32) yields the multi-monopole configuration with the operator

$$\hat{\rho}_0^{(n)} = \sum_{k=1}^{n} \hat{\rho}_{(k)0} = \sum_{k=1}^{n} (2R_k)^{-1} . \tag{5.42}$$

More general noncommutative Wu-Yang multi-monopoles may be inferred from appendixC.

6. Nonabelian Bogomolny-Prasad-Sommerfield monopoles

6.1 BPS monopole on commutative $\mathbb{R}^3$

Matrix-valued holomorphic function. We come to the main topic of the paper, the construction of noncommutative BPS monopoles via a Riemann-Hilbert problem. Let us first review the commutative case. It is well known that the spherically-symmetric SU(2) Bogomolny-Prasad-Sommerfield (BPS) monopole [20, 21, 48] permits a twistor description [24, 25, 26]. Its central object is the holomorphic matrix-valued function

$$f_{\mathrm{BPS}} = \begin{pmatrix} (\mathrm{e}^{w} - \mathrm{e}^{-w}) w^{-1} & \lambda^{-1}\mathrm{e}^{-w} \\ -\lambda\, \mathrm{e}^{-w} & w\, \mathrm{e}^{-w} \end{pmatrix} \tag{6.1}$$

which defines the corresponding Riemann-Hilbert problem. Here we abbreviate by w the combination

$$w := -\frac{\eta}{\lambda} = 2x^3 + \lambda\bar{y} - \lambda^{-1} y = (z + \lambda\bar{y}) - (-\bar{z} + \lambda^{-1} y) =: w_2 - \tilde{w}_1 , \tag{6.2}$$

where

$$w_2 = z + \lambda\bar{y} \qquad \text{and} \qquad \tilde{w}_1 = -\bar{z} + \lambda^{-1} y \tag{6.3}$$

for fixed $y, z, \bar{y}, \bar{z}$ are holomorphic on U_+ and U_-, respectively. Note that f_{BPS} does not depend on x^4 and satisfies the reality condition (3.16).

It is convenient to factorize the matrix (6.1) as follows (cf. [24]):

$$\begin{aligned} f_{\mathrm{BPS}} &= \begin{pmatrix} (\mathrm{e}^{w_2}\mathrm{e}^{-\tilde{w}_1} - \mathrm{e}^{-w_2}\mathrm{e}^{\tilde{w}_1}) w^{-1} & \lambda^{-1}\mathrm{e}^{-w_2}\mathrm{e}^{\tilde{w}_1} \\ -\lambda\, \mathrm{e}^{-w_2}\mathrm{e}^{\tilde{w}_1} & w\, \mathrm{e}^{-w_2}\mathrm{e}^{\tilde{w}_1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathrm{e}^{-w_2} \\ -\mathrm{e}^{w_2} & \lambda w \mathrm{e}^{-w_2} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ (\mathrm{e}^{2w_2} - \mathrm{e}^{2\tilde{w}_1}) w^{-1} & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\tilde{w}_1} & 0 \\ 0 & \mathrm{e}^{\tilde{w}_1} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{-w_2} & 0 \\ \lambda\, w\, \mathrm{e}^{-w_2} & \mathrm{e}^{w_2} \end{pmatrix} \begin{pmatrix} w^{-1}(\mathrm{e}^{2w_2} - \mathrm{e}^{2\tilde{w}_1}) & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\tilde{w}_1} & 0 \\ 0 & \mathrm{e}^{\tilde{w}_1} \end{pmatrix} . \end{aligned} \tag{6.4}$$

The left and right factors are holomorphic on U_+ and U_-, respectively, reducing our parametric Riemann-Hilbert problem to the middle factor, which takes the form

$$\begin{pmatrix} \rho & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} \qquad \text{with} \qquad \rho = w^{-1}(\mathrm{e}^{2w_2} - \mathrm{e}^{2\tilde{w}_1}) . \tag{6.5}$$

However, this type of matrix has already been split in the previous section (see also [29]). Therefore, we can proceed like for the Wu-Yang SU(2) monopole and expand ρ in a Laurent series in λ,

$$\rho = \sum_{m=-\infty}^{\infty} \rho_m \, \lambda^m = \rho_- + \rho_0 + \rho_+ \,, \tag{6.6}$$

where

$$\rho_+ := \sum_{m>0} \rho_m \, \lambda^m \qquad \text{and} \qquad \rho_- := \sum_{m<0} \rho_m \, \lambda^m \,. \tag{6.7}$$

There are two important differences from the previous section, however. First, our function ρ given by (6.5) does not satisfy the reality condition, i.e. $\rho(x, -1/\bar{\lambda}))^\dagger \neq \rho(x, \lambda)$. Second, it depends on x^4, but in a simple way. We can make this explicit by factorizing

$$\rho = \mathrm{e}^{-2\mathrm{i}x^4} \tilde{\rho} \,, \tag{6.8}$$

where

$$\tilde{\rho} := \frac{1}{w} \left(\mathrm{e}^{2x^3} \mathrm{e}^{2\lambda \bar{y}} - \mathrm{e}^{-2x^3} \mathrm{e}^{2\lambda^{-1} y} \right) = \tilde{\rho}_+ + \tilde{\rho}_0 + \tilde{\rho}_- \tag{6.9}$$

is a function of λ and of (x^1, x^2, x^3) only, with Laurent coefficients $\tilde{\rho}_m$.

Complex form of solution. Following the Wu-Yang case (cf. (5.18) and (5.19)) but multiplying the additional holomorphic matrix factors from (6.4), we split

$$f_{\mathrm{BPS}} = \psi_+^{-1} \psi_- \,, \tag{6.10}$$

where now

$$\begin{aligned}
\psi_- &= \rho_0^{-\frac{1}{2}} \begin{pmatrix} \rho_0 + \rho_- & \lambda^{-1} \\ \lambda \rho_- & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\tilde{w}_1} & 0 \\ 0 & \mathrm{e}^{\tilde{w}_1} \end{pmatrix} , \\
\psi_-^{-1} &= \begin{pmatrix} \mathrm{e}^{\tilde{w}_1} & 0 \\ 0 & \mathrm{e}^{-\tilde{w}_1} \end{pmatrix} \begin{pmatrix} 1 & -\lambda^{-1} \\ -\lambda \rho_- & \rho_0 + \rho_- \end{pmatrix} \rho_0^{-\frac{1}{2}} \,,
\end{aligned} \tag{6.11}$$

$$\begin{aligned}
\psi_+ &= \rho_0^{-\frac{1}{2}} \begin{pmatrix} 1 & \lambda^{-1} \rho_+ \\ \lambda & \rho_0 + \rho_+ \end{pmatrix} \begin{pmatrix} \mathrm{e}^{w_2} & 0 \\ -\lambda \mathrm{e}^{-w_2} w & \mathrm{e}^{-w_2} \end{pmatrix} , \\
\psi_+^{-1} &= \begin{pmatrix} \mathrm{e}^{-w_2} & 0 \\ \lambda w \mathrm{e}^{-w_2} & \mathrm{e}^{w_2} \end{pmatrix} \begin{pmatrix} \rho_0 + \rho_+ & -\lambda^{-1} \rho_+ \\ -\lambda & 1 \end{pmatrix} \rho_0^{-\frac{1}{2}} \,.
\end{aligned} \tag{6.12}$$

Formulae (6.10) – (6.12) solve a parametric Riemann-Hilbert problem on $\mathbb{C}P^1$ with external parameters $x \in \mathbb{R}^4$.

Having found $\psi_\pm(x, \lambda)$, one obtains a solution of the SDYM equations,

$$A_{\bar{y}} = \psi_+ \partial_{\bar{y}} \psi_+^{-1}|_{\lambda=0} \,, \quad A_{\bar{z}} = \psi_+ \partial_{\bar{z}} \psi_+^{-1}|_{\lambda=0} \,, \quad A_y = \psi_- \partial_y \psi_-^{-1}|_{\lambda=\infty} \,, \quad A_z = \psi_- \partial_z \psi_-^{-1}|_{\lambda=\infty} \,. \tag{6.13}$$

Substituting (6.11) and (6.12) into (6.13) and using the recursion relations

$$\partial_{\bar{y}} \, \rho_{m+1} = \partial_z \, \rho_m \qquad \text{and} \qquad \partial_{\bar{z}} \, \rho_{m+1} = -\partial_y \, \rho_m \tag{6.14}$$

implied by

$$(\partial_{\bar{y}} - \lambda\partial_z)\rho = (\partial_{\bar{z}} + \lambda\partial_y)\rho = 0\,, \tag{6.15}$$

we get in real coordinates

$$A_a = -\varepsilon_{abc}\,\frac{\sigma_c}{2\mathrm{i}}\,\tilde{\rho}_0^{-1}\partial_b\tilde{\rho}_0 + \sigma_a \qquad \text{and} \qquad \phi = -\frac{\sigma_a}{2\mathrm{i}}\,\tilde{\rho}_0^{-1}\partial_a\tilde{\rho}_0\,. \tag{6.16}$$

One sees that the above configuration (A_a, ϕ) (derived for the first time by Manton [49]) does not depend on x^4 (and therefore solves the Bogomolny equations) but is not real since $A_a^\dagger \neq -A_a$. However, Manton has shown by direct calculations that the solution (6.16) can be transformed by a complex gauge transformation to a real form coinciding with the standard BPS monopole [49] (see also [50, 51, 52]). The existence of such a gauge transformation follows from the fact that the matrix-valued function (6.1) producing (6.16) satisfies the reality condition (3.16).

Nonunitary gauge transformation to a real form of solution. The matrix g producing the above-mentioned transformation can be constructed as follows (cf. [53]). From the reality condition (3.16) for the Birkhoff decomposition $f_{\mathrm{BPS}} = \psi_+^{-1}\psi_-$ we obtain

$$\begin{aligned} &\left(\psi_+^{-1}(x,-\bar{\lambda}^{-1})\;\psi_-(x,-\bar{\lambda}^{-1})\right)^\dagger = \psi_+^{-1}(x,\lambda)\;\psi_-(x,\lambda) \qquad \Longrightarrow \\ &\Longrightarrow \qquad \psi_+(x,\lambda)\;\psi_-^\dagger(x,-\bar{\lambda}^{-1}) = \psi_-(x,\lambda)\;\psi_+^\dagger(x,-\bar{\lambda}^{-1}) =: g^2\,. \end{aligned} \tag{6.17}$$

Note that, for any fixed $x \in \mathbb{R}^4$, the product $\psi_+(x,\lambda)\psi_-^\dagger(x,-\bar{\lambda}^{-1})$ is holomorphic (in λ) on U_+ and $\psi_-(x,\lambda)\psi_+^\dagger(x,-\bar{\lambda}^{-1})$ is holomorphic on U_-. Therefore, g^2 does not depend on λ due to Liouville's theorem (globality on $\mathbb{C}P^1 = U_+ \cup U_-$). Moreover, we have

$$\psi_+^\dagger(x,-\bar{\lambda}^{-1}) = \psi_-^{-1}(x,\lambda)\,g^2 = \psi_+^\dagger(x,-\bar{\lambda}^{-1})\,(g^\dagger)^{-2}g^2 \qquad \Longrightarrow \qquad g^2 = (g^\dagger)^2\,. \tag{6.18}$$

If g^2 has no negative eigenvalues, it follows that $g = g^\dagger$, i.e. $g = g(x)$ is a hermitean matrix. One can calculate it for any λ in (6.17), e.g. by taking the square root of

$$g^2 = \psi_+(x,\lambda)\,\psi_-^\dagger(x,-\bar{\lambda}^{-1})|_{\lambda=0}\,. \tag{6.19}$$

Substituting $\psi_\pm$ from (6.11) and (6.12) into (6.19), we obtain

$$\begin{aligned} g^{-2} &= (\psi_-^\dagger(x,-\bar{\lambda}^{-1}))^{-1}\psi_+^{-1}(x,\lambda)|_{\lambda=0} = (\psi_+^\dagger(x,-\bar{\lambda}^{-1}))^{-1}\psi_-^{-1}(x,\lambda)|_{\lambda=\infty} \\ &= (\tilde{\rho}_0^\dagger)^{-\frac{1}{2}}\begin{pmatrix} \tilde{\rho}_0\mathrm{e}^{-2x^3} + \tilde{\rho}_{-1}^\dagger y\tilde{\rho}_0 & -\tilde{\rho}_1\mathrm{e}^{-2x^3} - \tilde{\rho}_{-1}^\dagger\mathrm{e}^{2x^3} - \tilde{\rho}_{-1}^\dagger y\tilde{\rho}_1 \\ -\tilde{\rho}_0^\dagger y\tilde{\rho}_0 & \tilde{\rho}_0\mathrm{e}^{2x^3} + \tilde{\rho}_0^\dagger y\tilde{\rho}_1 \end{pmatrix}\tilde{\rho}_0^{-\frac{1}{2}} \\ &= (\tilde{\rho}_0^\dagger)^{-\frac{1}{2}}\begin{pmatrix} \tilde{\rho}_0^\dagger\mathrm{e}^{-2x^3} + \tilde{\rho}_0^\dagger\bar{y}\tilde{\rho}_{-1} & -\tilde{\rho}_0\bar{y}\tilde{\rho}_0^\dagger \\ -\tilde{\rho}_1^\dagger\mathrm{e}^{-2x^3} - \tilde{\rho}_{-1}\mathrm{e}^{2x^3} - \tilde{\rho}_1^\dagger\bar{y}\tilde{\rho}_{-1} & \tilde{\rho}_0^\dagger\mathrm{e}^{2x^3} + \tilde{\rho}_1^\dagger\bar{y}\tilde{\rho}_0 \end{pmatrix}\tilde{\rho}_0^{-\frac{1}{2}}\,, \end{aligned} \tag{6.20}$$

which is x^4-independent as it should be.

After having taken the square root of (6.20) one can introduce

$$\psi_-^g := g^{-1}\psi_- \qquad \text{and} \qquad \psi_+^g := g^{-1}\psi_+\,, \tag{6.21}$$

which satisfy the reality condition (3.17),

$$\left(\psi_+^g(x,-\bar{\lambda}^{-1})\right)^\dagger = \psi_+^\dagger(x,-\bar{\lambda}^{-1})\,g^{-1} = \psi_-^{-1}(x,\lambda)\,g^2 g^{-1} = \left(\psi_-^g(x,\lambda)\right)^{-1} \tag{6.22}$$

due to eq. (6.18). By direct calculation one can show that the gauge-transformed fields

$$A_a^g = g^{-1}A_a g + g^{-1}\partial_a g \qquad \text{and} \qquad \phi^g = g^{-1}\phi\, g \tag{6.23}$$

are indeed antihermitean and thus represent a real form of our solution to the Bogomolny equations.

Explicit solution. It remains to compute the Laurent coefficients $\tilde{\rho}_0$ and $\tilde{\rho}_{\pm 1}$ of the function $\tilde{\rho} = w^{-1}(\mathrm{e}^{2x^3}\mathrm{e}^{2\lambda\bar{y}} - \mathrm{e}^{-2x^3}\mathrm{e}^{2\lambda^{-1}y})$ which enter the expressions (6.16) and (6.20). A point of concern may be that the denominator

$$w = w_2 - \tilde{w}_1 = \lambda\bar{y} + 2x^3 - \lambda^{-1}y = \frac{1}{\lambda}\,(\lambda-\lambda_1)\,(\lambda-\lambda_2)\,\bar{y} \tag{6.24}$$

of $\tilde{\rho}$ vanishes at points $\lambda_1, \lambda_2 \in \mathbb{C}P^1$ given by formulae (4.2). Nevertheless, the function $\tilde{\rho}$ is well defined at these points (and has poles only at $\lambda = 0$ and $\lambda = \infty$) as can be seen from its Laurent-series expansion.

From the experience with the Dirac and Wu-Yang monopoles we expect that the Riemann-Hilbert problem for the noncommutative BPS monopole, to be treated in the next subsection, takes the same form as eq. (6.4) modulo ordering ambiguities. Anticipating that the noncommutative cousin of ρ will appear in Weyl-ordered form, we shall organize the following (commutative) calculation in a symmetric manner which will remain valid in the noncommutative case (with coordinates promoted to operators). We begin by rewriting

$$\begin{aligned}\rho &= \mathrm{e}^{w_2}\,w^{-1}\,\mathrm{e}^{w_2} - \mathrm{e}^{\tilde{w}_1}\,w^{-1}\,\mathrm{e}^{\tilde{w}_1} \\ &= \mathrm{e}^{z+\lambda\bar{y}}\lambda\,\bar{y}^{-1}(\lambda-\lambda_2)^{-1}(\lambda-\lambda_1)^{-1}\mathrm{e}^{z+\lambda\bar{y}} - \mathrm{e}^{-z+\lambda^{-1}y}\lambda\,\bar{y}^{-1}(\lambda-\lambda_2)^{-1}(\lambda-\lambda_1)^{-1}\mathrm{e}^{-z+\lambda^{-1}y}\,.\end{aligned} \tag{6.25}$$

Making use of the relations

$$w_2(\lambda_1) = \tilde{w}_1(\lambda_1) \qquad \text{and} \qquad w_2(\lambda_2) = \tilde{w}_1(\lambda_2) \tag{6.26}$$

we decompose ρ into

$$\begin{aligned}\rho_+ &= \mathrm{e}^{w_2(\lambda)}\lambda\,\bar{y}^{-1}(\lambda-\lambda_2)^{-1}(\lambda-\lambda_1)^{-1}\mathrm{e}^{w_2(\lambda)} - \mathrm{e}^{w_2(\lambda_1)}\lambda\,\bar{y}^{-1}(\lambda_1-\lambda_2)^{-1}(\lambda-\lambda_1)^{-1}\mathrm{e}^{w_2(\lambda_1)} - \\ &\quad -\mathrm{e}^{w_2(\lambda_2)}\lambda\,\bar{y}^{-1}(\lambda-\lambda_2)^{-1}(\lambda_2-\lambda_1)^{-1}\mathrm{e}^{w_2(\lambda_2)}\,, \end{aligned}\tag{6.27}$$

$$\begin{aligned}\rho_- &= -\mathrm{e}^{\tilde{w}_1(\lambda)}\lambda\,\bar{y}^{-1}(\lambda-\lambda_2)^{-1}(\lambda-\lambda_1)^{-1}\mathrm{e}^{\tilde{w}_1(\lambda)} + \mathrm{e}^{\tilde{w}_1(\lambda_1)}\bar{y}^{-1}(\lambda_1-\lambda_2)^{-1}(\lambda-\lambda_1)^{-1}\lambda_1\mathrm{e}^{\tilde{w}_1(\lambda_1)} + \\ &\quad +\mathrm{e}^{\tilde{w}_1(\lambda_2)}\bar{y}^{-1}\lambda_2(\lambda-\lambda_2)^{-1}(\lambda_2-\lambda_1)^{-1}\mathrm{e}^{\tilde{w}_1(\lambda_2)}\,, \end{aligned}\tag{6.28}$$

$$\rho_0 = \mathrm{e}^{\tilde{w}_1(\lambda_1)}\bar{y}^{-1}(\lambda_1-\lambda_2)^{-1}\mathrm{e}^{\tilde{w}_1(\lambda_1)} + \mathrm{e}^{w_2(\lambda_2)}\bar{y}^{-1}(\lambda_2-\lambda_1)^{-1}\mathrm{e}^{w_2(\lambda_2)}\,. \tag{6.29}$$

Solving (6.24) for λ_1 and λ_2 as in (4.2) we obtain (for commuting quantities)

$$\lambda_1 = -\frac{r+x^3}{\bar{y}} = -\frac{y}{r-x^3} \qquad \text{and} \qquad \lambda_2 = \frac{y}{r+x^3} = \frac{r-x^3}{\bar{y}} \tag{6.30}$$

with $r^2 = y\bar{y} + x^3x^3$ and therefore

$$(\lambda_1{-}\lambda_2)\bar{y} = -2r\,, \qquad \tilde{w}_1(\lambda_1) = -\mathrm{i}x^4 - r\,, \qquad w_2(\lambda_2) = -\mathrm{i}x^4 + r \tag{6.31}$$

which simplifies (6.29) to

$$\rho_0 = \mathrm{e}^{-\mathrm{i}x^4-r}(-2r)^{-1}\mathrm{e}^{-\mathrm{i}x^4-r} + \mathrm{e}^{-\mathrm{i}x^4+r}(2r)^{-1}\mathrm{e}^{-\mathrm{i}x^4+r} \tag{6.32}$$

yielding

$$\tilde{\rho}_0 = \mathrm{e}^{2\mathrm{i}x^4}\,\rho_0 = \frac{\sinh(2r)}{r}\,. \tag{6.33}$$

Likewise, from (6.27) and (6.28) one can easily obtain

$$\tilde{\rho}_1 \;= \mathrm{e}^{2\mathrm{i}x^4}\lim_{\lambda\to 0}(\lambda^{-1}\rho_+) = -\frac{\mathrm{e}^{2x^3}}{y} + \frac{(r{-}x^3)}{y}\frac{\mathrm{e}^{-2r}}{2r} + \frac{(r{+}x^3)}{y}\frac{\mathrm{e}^{2r}}{2r}\,, \tag{6.34}$$

$$\tilde{\rho}_{-1} \;= \mathrm{e}^{2\mathrm{i}x^4}\lim_{\lambda\to\infty}(\lambda\,\rho_-) = -\frac{\mathrm{e}^{-2x^3}}{\bar{y}} + \frac{(r{+}x^3)}{\bar{y}}\frac{\mathrm{e}^{-2r}}{2r} + \frac{(r{-}x^3)}{\bar{y}}\frac{\mathrm{e}^{2r}}{2r}\,. \tag{6.35}$$

The simple form of $\tilde{\rho}_0$ immediately yields the complex form of the solution,

$$A_a = \varepsilon_{abc}\frac{\sigma_c}{2\mathrm{i}}\frac{x_b}{r}\left(\frac{1}{r} - 2\coth(2r)\right) + \sigma_a \qquad \text{and} \qquad \phi = \frac{\sigma_a}{2\mathrm{i}}\frac{x_a}{r}\left(\frac{1}{r} - 2\coth(2r)\right). \tag{6.36}$$

Next, we evaluate the nonunitary gauge matrix g. Inserting the functions $\tilde{\rho}_0$ and $\tilde{\rho}_{\pm 1}$ into (6.20) simplifies it to

$$g^2 = \mathrm{e}^{2x^a\sigma_a} \qquad \text{which allows for} \qquad g = \mathrm{e}^{x^a\sigma_a}\,. \tag{6.37}$$

Since this g commutes with ϕ we have $\phi^g = \phi$, but the gauge potential A_a transforms to an antihermitean A_a^g. The real form of the solution,

$$A_a^g = \varepsilon_{abc}\frac{\sigma_c}{2\mathrm{i}}\frac{x_b}{r}\left(\frac{1}{r} - \frac{2}{\sinh(2r)}\right) \qquad \text{and} \qquad \phi^g = \frac{\sigma_a}{2\mathrm{i}}\frac{x_a}{r}\left(\frac{1}{r} - 2\coth(2r)\right) \tag{6.38}$$

indeed coincides with the BPS monopole in its standard form.

6.2 Noncommutative BPS monopole

Some formulae. We are now ready to generalize the framework of the previous subsection to the noncommutative case. For the noncommutative space $\mathbb{R}^3_\theta$, we assume that a matrix-valued function encoding a charge one noncommutative U(2) BPS monopole still has the form (6.1) but with w being considered as an operator $\hat{w}$ acting in the one-oscillator Fock space $\mathcal{H}$. So, we define

$$\hat{w}_2 = z + \lambda\hat{\bar{y}} \qquad \text{and} \qquad \hat{\tilde{w}}_1 = \lambda^{-1}\hat{y} - \bar{z} \tag{6.39}$$

and

$$\hat{w} := \hat{w}_2 - \hat{\tilde{w}}_1 = z + \bar{z} + \lambda\hat{\bar{y}} - \lambda^{-1}\hat{y} = 2x^3 + \lambda\hat{\bar{y}} - \lambda^{-1}\hat{y}\,. \tag{6.40}$$

The commutation relations

$$[\,\hat{y}\,,\,\hat{\bar{y}}\,] = 2\theta \qquad \text{and} \qquad \text{other commutators} = 0 \qquad \Longrightarrow \qquad [\hat{\bar{w}}_1, \hat{w}_2] = 2\theta \tag{6.41}$$

imply that

$$\hat{w}_2\,\mathrm{e}^{\alpha\hat{\bar{w}}_1} = \mathrm{e}^{\alpha\hat{\bar{w}}_1}(\hat{w}_2 - 2\alpha\theta) \qquad \text{and} \qquad \hat{\bar{w}}_1\,\mathrm{e}^{\alpha\hat{w}_2} = \mathrm{e}^{\alpha\hat{w}_2}(\hat{\bar{w}}_1 + 2\alpha\theta)\ , \tag{6.42}$$

$$\hat{w}\,\mathrm{e}^{\alpha\hat{\bar{w}}_1} = \mathrm{e}^{\alpha\hat{\bar{w}}_1}(\hat{w} - 2\alpha\theta) \qquad \text{and} \qquad \hat{w}\,\mathrm{e}^{\alpha\hat{w}_2} = \mathrm{e}^{\alpha\hat{w}_2}(\hat{w} - 2\alpha\theta)\ , \tag{6.43}$$

$$\mathrm{e}^{\pm\hat{w}} = \mathrm{e}^{-\theta}\,\mathrm{e}^{\pm\hat{w}_2}\,\mathrm{e}^{\mp\hat{\bar{w}}_1} = \mathrm{e}^{\theta}\,\mathrm{e}^{\mp\hat{\bar{w}}_1}\,\mathrm{e}^{\pm\hat{w}_2}\,, \tag{6.44}$$

where α is an arbitrary constant. Also we have

$$(\hat{\bar{w}}_1(-\bar{\lambda}^{-1}))^\dagger = -\hat{w}_2(\lambda)\,, \qquad (\hat{w}_2(-\bar{\lambda}^{-1}))^\dagger = -\hat{\bar{w}}_1(\lambda) \qquad \Longrightarrow \qquad (\hat{w}(-\bar{\lambda}^{-1}))^\dagger = \hat{w}(\lambda)\,. \tag{6.45}$$

Matrix-valued operator. Repeating arguments from section 4.2, we admit in (6.40) only such λ that the operator $\hat{w}$ will not have zero modes on the Hilbert spaces $\mathcal{H}$ or $\mathcal{H}^*$. As was shown in section 4.2, the operator $\hat{w}$ is invertible iff $|\lambda| = 1$. In the following we consider the operator-valued matrix

$$\hat{f}_{\mathrm{BPS}} = \begin{pmatrix} (\mathrm{e}^{\hat{w}} - \mathrm{e}^{-\hat{w}})\hat{w}^{-1} & \lambda^{-1}\mathrm{e}^{-\hat{w}} \\ -\lambda\,\mathrm{e}^{-\hat{w}} & \hat{w}\,\mathrm{e}^{-\hat{w}} \end{pmatrix} \tag{6.46}$$

restricted to $S^1 = \{\lambda \in \mathbb{C}P^1 : |\lambda| = 1\}$, where $\hat{y}$, $\hat{\bar{y}}$ and x^3 are considered as external "parameters". The calculations in the noncommutative case can almost literally be copied from the commutative situation treated in the previous subsection.

Splitting. The operator-valued matrix (6.46) is split as follows,

$$\begin{aligned}
\mathrm{e}^{-\theta}\hat{f}_{\mathrm{BPS}} &= \begin{pmatrix} (\mathrm{e}^{\hat{w}_2}\mathrm{e}^{-\hat{\bar{w}}_1} - \mathrm{e}^{-\hat{w}_2}\mathrm{e}^{\hat{\bar{w}}_1})\hat{w}^{-1} & \lambda^{-1}\mathrm{e}^{-\hat{w}_2}\mathrm{e}^{\hat{\bar{w}}_1} \\ -\lambda\,\mathrm{e}^{-\hat{w}_2}\mathrm{e}^{\hat{\bar{w}}_1} & \hat{w}\,\mathrm{e}^{-\hat{w}_2}\mathrm{e}^{\hat{\bar{w}}_1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathrm{e}^{-\hat{w}_2} \\ -\mathrm{e}^{\hat{w}_2} & \lambda\hat{w}\mathrm{e}^{-\hat{w}_2} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ (\mathrm{e}^{2\hat{w}_2} - \mathrm{e}^{2\hat{\bar{w}}_1})(\hat{w} - 2\theta)^{-1} & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\hat{\bar{w}}_1} & 0 \\ 0 & \mathrm{e}^{\hat{\bar{w}}_1} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{-\hat{w}_2} & 0 \\ \lambda\,\hat{w}\,\mathrm{e}^{-\hat{w}_2} & \mathrm{e}^{\hat{w}_2} \end{pmatrix} \begin{pmatrix} (\hat{w} + 2\theta)^{-1}(\mathrm{e}^{2\hat{w}_2} - \mathrm{e}^{2\hat{\bar{w}}_1}) & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\hat{\bar{w}}_1} & 0 \\ 0 & \mathrm{e}^{\hat{\bar{w}}_1} \end{pmatrix} \\
&= \hat{\psi}_+^{-1}\,\hat{\psi}_-\,,
\end{aligned} \tag{6.47}$$

where

$$\begin{aligned}
\hat{\psi}_- &= \hat{\rho}_0^{-\frac{1}{2}} \begin{pmatrix} \hat{\rho}_0 + \hat{\rho}_- & \lambda^{-1} \\ \lambda\hat{\rho}_- & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\hat{\bar{w}}_1} & 0 \\ 0 & \mathrm{e}^{\hat{\bar{w}}_1} \end{pmatrix}, \\
\hat{\psi}_-^{-1} &= \begin{pmatrix} \mathrm{e}^{\hat{\bar{w}}_1} & 0 \\ 0 & \mathrm{e}^{-\hat{\bar{w}}_1} \end{pmatrix} \begin{pmatrix} 1 & -\lambda^{-1} \\ -\lambda\hat{\rho}_- & \hat{\rho}_0 + \hat{\rho}_- \end{pmatrix} \hat{\rho}_0^{-\frac{1}{2}}\,,
\end{aligned} \tag{6.48}$$

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$$
\begin{aligned}
\hat{\psi}_+ &= \hat{\rho}_0^{-\frac{1}{2}} \begin{pmatrix} 1 & \lambda^{-1}\hat{\rho}_+ \\ \lambda & \hat{\rho}_0{+}\hat{\rho}_+ \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\hat{w}_2} & 0 \\ -\lambda \mathrm{e}^{-\hat{w}_2}\hat{w} & \mathrm{e}^{-\hat{w}_2} \end{pmatrix}, \\
\hat{\psi}_+^{-1} &= \begin{pmatrix} \mathrm{e}^{-\hat{w}_2} & 0 \\ \lambda\hat{w}\mathrm{e}^{-\hat{w}_2} & \mathrm{e}^{\hat{w}_2} \end{pmatrix} \begin{pmatrix} \hat{\rho}_0{+}\hat{\rho}_+ & -\lambda^{-1}\hat{\rho}_+ \\ -\lambda & 1 \end{pmatrix} \hat{\rho}_0^{-\frac{1}{2}}
\end{aligned} \tag{6.49}
$$

take the same form as in the commutative case (cf. (6.11) and (6.12)), and

$$
\begin{aligned}
\hat{\rho} &:= (\hat{w}{+}2\theta)^{-1}(\mathrm{e}^{2\hat{w}_2} - \mathrm{e}^{2\hat{\hat{w}}_1}) = \mathrm{e}^{\hat{w}_2}\hat{w}^{-1}\mathrm{e}^{\hat{w}_2} - \mathrm{e}^{\hat{\hat{w}}_1}\hat{w}^{-1}\mathrm{e}^{\hat{\hat{w}}_1} \\
&= \int_{-1}^{1} \mathrm{d}t\ \mathrm{e}^{(1+t)\hat{w}_2+(1-t)\hat{\hat{w}}_1} = \mathrm{e}^{2\mathrm{i}x^4} \int_{-1}^{1} \mathrm{d}t\ \mathrm{e}^{2tx^3}\ \mathrm{e}^{(1+t)\lambda\hat{\hat{y}}+(1-t)\lambda^{-1}\hat{y}} \\
&= \hat{\rho}_+ + \hat{\rho}_0 + \hat{\rho}_- = \mathrm{e}^{-2\mathrm{i}x^4}\,(\hat{\hat{\rho}}_+ + \hat{\hat{\rho}}_0 + \hat{\hat{\rho}}_-) = \mathrm{e}^{-2\mathrm{i}x^4}\,\hat{\hat{\rho}}
\end{aligned} \tag{6.50}
$$

is indeed Weyl ordered. Notice that also the operators $\hat{w} \pm 2\theta$ have no zero modes on $\mathcal{H}$ or $\mathcal{H}^*$ if $|\lambda| = 1$.

Explicit form of $\hat{\hat{\rho}}_0$ and $\hat{\hat{\rho}}_{\pm 1}$. For the computation of $\hat{\hat{\rho}}$ we may go back to the commutative case and simply put hats in eqs. (6.25)–(6.29), including over λ_1 and λ_2 which implicitly depend on $\hat{y}$ and $\hat{\hat{y}}$. However, to proceed from the noncommutative extension of (6.29) one cannot employ the expression (6.30) for the zeros $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of $\hat{w}$. Instead, we use the factorization (4.30) introduced for the noncommutative Dirac monopole,

$$
\hat{w} = -(\xi + \lambda^{-1}\hat{y}\xi^{-1})\,(\xi - \lambda\xi^{-1}\hat{\hat{y}}) \qquad \text{with} \qquad \xi^2 - \hat{y}\xi^{-2}\hat{\hat{y}} = -2x^3\,, \tag{6.51}
$$

which involves the hermitean operator ξ. Comparison with (6.24) implies[7]

$$
\hat{\lambda}_1 = -\hat{y}\,\xi^{-2} \qquad \text{and} \qquad \hat{\lambda}_2\,\hat{\hat{y}} = \xi^2 \qquad \text{hence} \qquad \hat{\lambda}_2 = \xi^2\,(\hat{y}\hat{\hat{y}})^{-1}\hat{y}\,. \tag{6.52}
$$

Defining again $R := x^3 + \xi^2$, we may write

$$
(\hat{\lambda}_1{-}\hat{\lambda}_2) = -2R\,(\hat{y}\hat{\hat{y}})^{-1}\hat{y} \qquad \text{and} \qquad \hat{\lambda}_1\,\hat{\lambda}_2 = -\hat{y}\,(\hat{y}\hat{\hat{y}})^{-1}\hat{y} \tag{6.53}
$$

and also

$$
\hat{\hat{w}}_1(\hat{\lambda}_1) = \hat{w}_2(\hat{\lambda}_1) = -\mathrm{i}x^4 - R \qquad \text{and} \qquad \hat{\hat{w}}_1(\hat{\lambda}_2) = \hat{w}_2(\hat{\lambda}_2) = -\mathrm{i}x^4 + R\,. \tag{6.54}
$$

Employing these relations in (the noncommutative version of) (6.29) we get

$$
\hat{\rho}_0 = \mathrm{e}^{-\mathrm{i}x^4-R}(-2R)^{-1}\mathrm{e}^{-\mathrm{i}x^4-R} + \mathrm{e}^{-\mathrm{i}x^4+R}(2R)^{-1}\mathrm{e}^{-\mathrm{i}x^4+R} \tag{6.55}
$$

which means

$$
\hat{\hat{\rho}}_0 = \frac{\sinh(2R)}{R} = \hat{\hat{\rho}}_0^{\dagger}\,. \tag{6.56}
$$

Analogously, with the help of a noncommutative variant of (6.30),

$$
\hat{y}\,(R{-}x^3)^{-1} = (R{+}x^3)\,(\hat{y}\hat{\hat{y}})^{-1}\hat{y} \qquad \text{and} \qquad (R{-}x^3)^{-1}\hat{\hat{y}} = \hat{y}\,(\hat{y}\hat{\hat{y}})^{-1}(R{+}x^3)\,, \tag{6.57}
$$

[7] We choose $\hat{\lambda}_2|0\rangle = 0$.

and the relations

$$1 = \hat{y}\,\hat{\bar{y}}\,(\hat{y}\hat{\bar{y}})^{-1} = \hat{y}\,(\hat{\bar{y}}\hat{y})^{-1}\,\hat{\bar{y}} = (\hat{y}\hat{\bar{y}})^{-1}\,\hat{y}\,\hat{\bar{y}} \tag{6.58}$$

we extract from (6.27) and (6.28) the Laurent coefficients

$$\hat{\tilde{\rho}}_1 = -\hat{\bar{y}}\,(\hat{y}\hat{\bar{y}})^{-1}\mathrm{e}^{2x^3} + (2R)^{-1}(R{-}x^3)\,\mathrm{e}^{-R}\hat{\bar{y}}\,(\hat{y}\hat{\bar{y}})^{-1}\mathrm{e}^{-R} + \mathrm{e}^{R}\,\hat{\bar{y}}\,(\hat{y}\hat{\bar{y}})^{-1}\mathrm{e}^{R}(R{+}x^3)(2R)^{-1}\ , \tag{6.59}$$

$$\hat{\tilde{\rho}}_{-1} = -(\hat{y}\hat{\bar{y}})^{-1}\hat{y}\,\mathrm{e}^{-2x^3} + (2R)^{-1}(R{+}x^3)\,\mathrm{e}^{-R}(\hat{y}\hat{\bar{y}})^{-1}\hat{y}\,\mathrm{e}^{-R} + \mathrm{e}^{R}(\hat{y}\hat{\bar{y}})^{-1}\hat{y}\,\mathrm{e}^{R}(R{-}x^3)(2R)^{-1}\,. \tag{6.60}$$

We notice a formal similarity with the commutative result (6.33)–(6.35), which becomes concrete in the commutative limit (cf. (4.7)),

$$\xi^2 \;\to\; \xi_+^2 = r - x^3 \qquad \text{hence} \qquad R \;\to\; r\,. \tag{6.61}$$

Noncommutative BPS monopole in a complex gauge. Substituting (6.56) into (3.22) and using the recursion relations[8] (6.14) for the Laurent coefficient $\hat{\rho}_m$ of $\hat{\rho}$, we get (cf. eq. (3.20) in [29])

$$\hat{A}_\mu \;=\; \bar{\eta}^a_{\mu\nu}\,\frac{\sigma_a}{2\mathrm{i}}\Big(\hat{\rho}_0^{\frac12}\,\partial_\nu\hat{\rho}_0^{-\frac12} - \hat{\rho}_0^{-\frac12}\partial_\nu\hat{\rho}_0^{\frac12}\Big) + \frac{\mathbf{1}_2}{2}\Big(\hat{\rho}_0^{-\frac12}\partial_\mu\hat{\rho}_0^{\frac12} + \hat{\rho}_0^{\frac12}\,\partial_\mu\hat{\rho}_0^{-\frac12}\Big) \tag{6.62}$$

$$\begin{aligned}
\Longrightarrow\qquad \hat{A}_a \;&=\; \varepsilon_{abc}\,\frac{\sigma_c}{2\mathrm{i}}\Big(\hat{\tilde{\rho}}_0^{\frac12}\,\partial_b\hat{\tilde{\rho}}_0^{-\frac12} - \hat{\tilde{\rho}}_0^{-\frac12}\partial_b\hat{\tilde{\rho}}_0^{\frac12}\Big) + \frac{\mathbf{1}_2}{2}\Big(\hat{\tilde{\rho}}_0^{-\frac12}\partial_a\hat{\tilde{\rho}}_0^{\frac12} + \hat{\tilde{\rho}}_0^{\frac12}\,\partial_a\hat{\tilde{\rho}}_0^{-\frac12}\Big) + \sigma_a\,, \\
\hat{\phi} \equiv \hat{A}_4 \;&=\; \frac{\sigma_a}{2\mathrm{i}}\Big(\hat{\tilde{\rho}}_0^{\frac12}\,\partial_a\hat{\tilde{\rho}}_0^{-\frac12} - \hat{\tilde{\rho}}_0^{-\frac12}\partial_a\hat{\tilde{\rho}}_0^{\frac12}\Big)\,,
\end{aligned} \tag{6.63}$$

where the operator $\hat{\tilde{\rho}}_0$ is given by (6.56). It is interesting that $\hat{\tilde{\rho}}_0$ is expressed via the operator ξ^2 which defines the noncommutative Dirac monopole (see section 4.2). One sees that the configuration $(\hat{A}_a, \hat{\phi})$ does not depend on x^4. We have obtained a solution of the noncommutative Bogomolny equations which is not real since $\hat{A}_a^\dagger \neq -\hat{A}_a$ analogously to the commutative case. Thus, (6.63) is the noncommutative BPS monopole written in a complex gauge.[9]

Gauge transformation to a real configuration. Similar to the commutative case one can derive an operator-valued matrix $\hat{g}(\hat{x})$ which generates a gauge transformation to a real form of the noncommutative BPS monopole. In fact, the formulae (6.17) – (6.19) for the above-mentioned matrix and the gauge transformations (6.21) and (6.23) apply literally to the noncommutative setup. The explicit form of $\hat{g}^{-2}$ in terms of $\hat{\tilde{\rho}}_0$ and $\hat{\tilde{\rho}}_{\pm1}$ is, however, more involved than in the commutative situation:

$$\begin{aligned}
\hat{g}^{-2} \;&=\; (\hat{\psi}_-^\dagger(\hat{x},-\bar{\lambda}^{-1}))^{-1}\,\hat{\psi}_+^{-1}(\hat{x},\lambda)|_{\lambda=0} = (\hat{\psi}_+^\dagger(\hat{x},-\bar{\lambda}^{-1}))^{-1}\,\hat{\psi}_-^{-1}(\hat{x},\lambda)|_{\lambda=\infty} \\
&=\; (\hat{\tilde{\rho}}_0^\dagger)^{-\frac12}\begin{pmatrix} \hat{\tilde{\rho}}_0\mathrm{e}^{-2x^3} + \hat{\tilde{\rho}}_{-1}^\dagger\hat{y}\hat{\tilde{\rho}}_0 & -\hat{\tilde{\rho}}_1\mathrm{e}^{-2x^3} - \hat{\tilde{\rho}}_{-1}^\dagger\mathrm{e}^{2x^3} - \hat{\tilde{\rho}}_{-1}^\dagger\hat{y}\hat{\tilde{\rho}}_1 \\ -\hat{\tilde{\rho}}_0^\dagger\hat{y}\hat{\tilde{\rho}}_0 & \hat{\tilde{\rho}}_0\mathrm{e}^{2x^3} + \hat{\tilde{\rho}}_0^\dagger\hat{y}\hat{\tilde{\rho}}_1 \end{pmatrix}\hat{\tilde{\rho}}_0^{-\frac12}
\end{aligned}$$

[8]These recursion relations keep their form in the noncommutative case.

[9]Note, however, that $\hat{\phi}$ is already antihermitean.

$$= (\hat{\hat{\rho}}_0^\dagger)^{-\frac{1}{2}} \begin{pmatrix} \hat{\hat{\rho}}_0^\dagger \mathrm{e}^{-2x^3} + \hat{\hat{\rho}}_0^\dagger \hat{\bar{y}} \hat{\hat{\rho}}_{-1} & -\hat{\hat{\rho}}_0 \hat{\bar{y}} \hat{\hat{\rho}}_0^\dagger \\ -\hat{\hat{\rho}}_1^\dagger \mathrm{e}^{-2x^3} - \hat{\hat{\rho}}_{-1} \mathrm{e}^{2x^3} - \hat{\hat{\rho}}_1^\dagger \hat{\bar{y}} \hat{\hat{\rho}}_{-1} & \hat{\hat{\rho}}_0^\dagger \mathrm{e}^{2x^3} + \hat{\hat{\rho}}_1^\dagger \hat{y} \hat{\hat{\rho}}_0 \end{pmatrix} \hat{\hat{\rho}}_0^{-\frac{1}{2}}$$

$$= (\hat{\hat{\rho}}_0)^{-\frac{1}{2}} \begin{pmatrix} \hat{\hat{\rho}}_0 (\mathrm{e}^{-2x^3} + \hat{\bar{y}} \hat{\hat{\rho}}_{-1}) & -\hat{\hat{\rho}}_0 \hat{\bar{y}} \hat{\hat{\rho}}_0 \\ -\hat{\hat{\rho}}_0 \hat{y} \hat{\hat{\rho}}_0 & \hat{\hat{\rho}}_0 (\mathrm{e}^{2x^3} + \hat{y} \hat{\hat{\rho}}_{-1}) \end{pmatrix} \hat{\hat{\rho}}_0^{-\frac{1}{2}} , \tag{6.64}$$

where we used the hermiticity of $\hat{\hat{\rho}}_0$ as given by (6.56). After inserting the expressions (6.56), (6.59) and (6.60) for $\hat{\hat{\rho}}_0$, $\hat{\hat{\rho}}_1$ and $\hat{\hat{\rho}}_{-1}$, respectively, it is unlikely that the matrix (6.64) can be converted into a nice exponential form (cf. (6.37) in the commutative case) which would permit us to explicitly take the square root $\hat{g}$ of $\hat{g}^2$ and insert into the formulae (6.21) and (6.23). Therefore, we must be content with an implicit real form of the noncommutative BPS monopole, given by (6.63) and (6.64).

Noncommutative BPS multi-monopoles may be treated in the same spirit, by employing matrices more general than (6.46) which are known for the commutative case [25, 26].

7. Conclusions

In this paper we have developed a unified approach to noncommutative monopoles of Dirac, Wu-Yang and BPS type, by reformulating their construction as a parametric Riemann-Hilbert problem. Although this method is well known in the commutative situation, its extension to the noncommutative case seems to be new, and it is equally fruitful.[10] In fact, most steps — writing the Bogomolny equation, formulating the auxiliary linear system, posing the Riemann-Hilbert problem for a suitably chosen matrix-valued function and even splitting the latter — can be generalized without much difficulty. This was demonstrated here by treating the commutative and noncommutative construction in succession for each type of monopole. In particular, the noncommutative Dirac and U(2) BPS monopoles were rederived, and explicit noncommutative U(2) Wu-Yang multi-monopoles were constructed for the first time.

Compared to previous derivations of noncommutative monopole configurations, our presentation is just as practical in calculations as it is systematic. For instance, the construction of the noncommutative Dirac monopole [7] reduces to a few lines of algebra, and the key relation (see (7.3) below) emerges naturally. Of course, some of the computational details are more intricate than in the commutative case. Let us mention two points here for the example of the noncommutative U(2) BPS monopole. First, the Riemann-Hilbert problem generically yields the monopole potential in a complex gauge (which we constructed explicitly even in the noncommutative case), but it also provides one with a recipe for gauge transforming to a real configuration. Yet, the requisite nonunitary gauge transformation seems to simplify only in the commutative limit. Second, it is not the factorization of the 2×2 matrix $\hat{f}_{\mathrm{BPS}}$ but the decomposition of its nontrivial entry $\hat{\rho}$ which requires more skill in the noncommutative setup.

[10] It works even in the framework of string field theory [54].

As a technical remark, in all cases considered in this paper we are led to consider for fixed x the zeros λ_1 and λ_2 of the function

$$\eta(x,\lambda) = y - 2\lambda x^3 - \lambda^2\bar{y} = (\lambda\xi_+ + y\xi_+^{-1})\,(\xi_+ - \lambda\xi_+^{-1}\bar{y}) \tag{7.1}$$

which gets deformed noncommutatively to

$$\hat{\eta}(\hat{x},\lambda) = \hat{y} - 2\lambda x^3 - \lambda^2\hat{\bar{y}} = (\lambda\xi + \hat{y}\xi^{-1})\,(\xi - \lambda\xi^{-1}\hat{\bar{y}})\,. \tag{7.2}$$

Here, the operator ξ is defined by

$$\xi^2 - \hat{y}\xi^{-2}\hat{\bar{y}} + 2x^3 = 0 \tag{7.3}$$

and in the commutative limit tends to the function

$$\xi_+ = (r - x^3)^{1/2} \qquad \text{with} \quad r^2 = y\bar{y} + x^3x^3\,. \tag{7.4}$$

Since the deformed variable $\hat{\eta}$ still commutes with λ, the matrix-valued operators $\hat{f}(\hat{\eta},\lambda)$ continue to look like matrix-valued *functions* in the noncommutative setup.

While formulating Riemann-Hilbert problems for (commutative or noncommutative) *multi*-monopoles is straightforward, solving them is another story. We succeeded here for a certain class of Wu-Yang multi-monopoles but must otherwise leave this task for future research.

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A. Self-dual gauge fields in the twistor approach

SDYM equations. It is well known (see e.g. [49, 41, 55]) that the Bogomolny equations (2.9) are equivalent to the reduced self-dual Yang-Mills (SDYM) equations in four euclidean dimensions. Namely, let us consider the euclidean space $\mathbb{R}^4$ with the metric $\delta_{\mu\nu}$, a gauge potential $A = A_\mu \mathrm{d}x^\mu$ and the Yang-Mills field $F = \mathrm{d}A + A\wedge A = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ where $\mu,\nu,\ldots = 1,2,3,4$. We assume that the fields A and F take values in the Lie algebra $u(n)$.

The SDYM equations have the form

$$*F = F \qquad \Longrightarrow \qquad \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} = F_{\mu\nu}\,, \tag{A.1}$$

where $*$ denotes the Hodge star operator and $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor in $\mathbb{R}^4$, with $\varepsilon_{1234} = 1$. If one assumes that the A_μ do not depend on x^4 and puts $\phi := A_4$

then (A.1) reduces to the Bogomolny equations (2.9). Moreover, under the same conditions the energy density in (2.8) follows from the pure Yang-Mills lagrangian density

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu}) \tag{A.2}$$

in four euclidean dimensions.

As reduction of the SDYM equations, the Bogomolny equations are seen to be integrable. Hence, one may apply powerful twistor methods, which have been developed for solving the SDYM equations, also to the construction of monopole solutions. Therefore, we briefly recall the twistor description of self-dual gauge fields in this appendix.

Linear system. Let us introduce the complex coordinates

$$y = x^1 + \mathrm{i}x^2\,, \qquad z = x^3 - \mathrm{i}x^4\,, \qquad \bar{y} = x^1 - \mathrm{i}x^2\,, \qquad \bar{z} = x^3 + \mathrm{i}x^4 \tag{A.3}$$

and put

$$A_y = \frac{1}{2}(A_1 - \mathrm{i}A_2)\,, \qquad A_z = \frac{1}{2}(A_3 + \mathrm{i}A_4)\,, \qquad A_{\bar{y}} = \frac{1}{2}(A_1 + \mathrm{i}A_2)\,, \qquad A_{\bar{z}} = \frac{1}{2}(A_3 - \mathrm{i}A_4)\,. \tag{A.4}$$

Then the SDYM equations (A.1) read

$$[D_y, D_z] = 0\,, \qquad [D_{\bar{y}}, D_{\bar{z}}] = 0\,, \qquad [D_y, D_{\bar{y}}] + [D_z, D_{\bar{z}}] = 0\,, \tag{A.5}$$

where $D_\mu := \partial_\mu + A_\mu$.

These are equations for the connection 1-form $A = A_\mu \mathrm{d}x^\mu$ on a (trivial) complex vector bundle

$$E = \mathbb{R}^4 \times \mathbb{C}^n \tag{A.6}$$

over $\mathbb{R}^4$ associated to the principal bundle $P = \mathbb{R}^4 \times \mathrm{U}(n)$. They can be obtained as the compatibility conditions of the following linear system of equations [33, 34]:

$$(D_{\bar{y}} - \lambda D_z)\,\psi(x,\lambda,\bar{\lambda}) = 0\,, \qquad (D_{\bar{z}} + \lambda D_y)\,\psi(x,\lambda,\bar{\lambda}) = 0\,, \quad \text{and} \quad \partial_{\bar{\lambda}}\psi(x,\lambda,\bar{\lambda}) = 0\,, \tag{A.7}$$

where the $n \times n$ matrix ψ depends on an extra 'spectral parameter' λ which lies in the extended complex plane $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$.

Twistor space. In fact, the matrix-valued function ψ in (A.7) is defined on the *twistor space* $\mathcal{Z} = \mathbb{R}^4 \times S^2$ for the space $\mathbb{R}^4$ [45, 56, 23] with the canonical projection

$$\pi \;:\; \mathcal{Z} \;\to\; \mathbb{R}^4\,. \tag{A.8}$$

It can be viewed as a collection of n sections of the pulled-back vector bundle π^*E over $\mathcal{Z}$. Then, as we shall explain shortly, (A.7) is interpreted as the holomorphicity of the vector bundle $\pi^*E \to \mathcal{Z}$.

Recall that the Riemann sphere $S^2 \cong \mathbb{C}P^1$ can be covered by two coordinate patches U_+ and U_- with

$$\mathbb{C}P^1 = U_+ \cup U_-\,, \qquad U_+ = \mathbb{C}P^1 \setminus \{\infty\}\,, \qquad U_- = \mathbb{C}P^1 \setminus \{0\}\,, \tag{A.9}$$

and coordinates λ and $\tilde{\lambda}$ on U_+ and U_-, respectively. Therefore, also $\mathcal{Z}$ can be covered by two coordinate patches,

$$\mathcal{Z} = \mathcal{U}_+ \cup \mathcal{U}_- , \qquad \mathcal{U}_+ = \mathbb{R}^4 \times U_+ , \qquad \mathcal{U}_- = \mathbb{R}^4 \times U_- , \tag{A.10}$$

with coordinates $\{x^\mu, \lambda, \bar{\lambda}\}$ on $\mathcal{U}_+$ and $\{x^\mu, \tilde{\lambda}, \bar{\tilde{\lambda}}\}$ on $\mathcal{U}_-$. One can also introduce complex coordinates

$$w_1 = y - \lambda\bar{z} , \quad w_2 = z + \lambda\bar{y} , \quad w_3 = \lambda \quad \text{and} \quad \tilde{w}_1 = \tilde{\lambda}y - \bar{z} , \quad \tilde{w}_2 = \tilde{\lambda}z + \bar{y} , \quad \tilde{w}_3 = \tilde{\lambda} \tag{A.11}$$

on $\mathcal{U}_+$ and $\mathcal{U}_-$, respectively. On the intersection $\mathcal{U}_+ \cap \mathcal{U}_- \cong \mathbb{R}^4 \times \mathbb{C}^*$ these coordinates are related by

$$\tilde{w}_1 = \frac{w_1}{w_3} , \qquad \tilde{w}_2 = \frac{w_2}{w_3} \qquad \text{and} \qquad \tilde{w}_3 = \frac{1}{w_3} . \tag{A.12}$$

On the open set $\mathcal{U}_+ \cap \mathcal{U}_-$ one may use either set of coordinates, and we will use w_1, w_2 and w_3.

Complex and real structures on $\mathcal{Z}$. It follows from (A.12) that, as a complex manifold, $\mathcal{Z}$ is not a direct product $\mathbb{C}^2 \times \mathbb{C}P^1$ but is a nontrivial rank two holomorphic vector bundle over $\mathbb{C}P^1$ with a holomorphic projection

$$p \; : \quad \mathcal{Z} = \mathcal{O}(1) \otimes \mathbb{C}^2 = \mathcal{O}(1) \oplus \mathcal{O}(1) \;\; \to \;\; \mathbb{C}P^1 , \tag{A.13}$$

where $\mathcal{O}(k)$ is the holomorphic line bundle over $\mathbb{C}P^1$ with first Chern class $c_1(\mathcal{O}(k)) = k$. Real holomorphic sections of the bundle (A.13) are the projective lines

$$\mathbb{C}P^1_x = \begin{cases} \lambda \in U_+ , & w_1 = y - \lambda\bar{z} , \quad w_2 = z + \lambda\bar{y} \\ \lambda \in U_- , & \tilde{w}_1 = \tilde{\lambda}y - \bar{z} , \quad \tilde{w}_2 = \tilde{\lambda}z + \bar{y} \end{cases} \tag{A.14}$$

parametrized by points $x \in \mathbb{R}^4$. These sections are real in the sense that they are invariant w.r.t. an antiholomorphic involution (*real structure*) $\tau : \; \mathcal{Z} \to \mathcal{Z}$ defined by

$$\tau(y, \bar{y}, z, \bar{z}, \lambda) = (\bar{y}, y, \bar{z}, z, -\bar{\lambda}^{-1}) . \tag{A.15}$$

It is easily seen from (A.15) that τ has no fixed points in $\mathcal{Z}$ but does leave the projective lines $\mathbb{C}P^1_x \hookrightarrow \mathcal{Z}$ invariant. These Riemann spheres $\mathbb{C}P^1_x$ are parametrized by $(x^\mu) = (y, \bar{y}, z, \bar{z}) \in \mathbb{R}^4$. Note that on matrix-valued functions $\varphi(x, \lambda)$ the involution τ acts as follows [57]:

$$\tau \;\; : \quad \varphi(x, \lambda) \mapsto (\tau\varphi)(x, \lambda) := [\varphi(x, -\bar{\lambda}^{-1})]^\dagger . \tag{A.16}$$

Pulled-back bundle $\pi^* E$. Consider the complex vector bundle E introduced in (A.6) and its pullback $\pi^* E$ to $\mathcal{Z}$. Note that the fibres of the bundle (A.8) over points $x \in \mathbb{R}^4$ are the two-spheres (A.14), i.e. $\pi^{-1}(x) = \mathbb{C}P^1_x$. By definition, the pulled-back bundle $\pi^* E$ is trivial on the fibres $\mathbb{C}P^1_x$. For the pulled-back connection $\tilde{D} := \pi^* D = D + \mathrm{d}\lambda\partial_\lambda + \mathrm{d}\bar{\lambda}\partial_{\bar{\lambda}}$ one can introduce the $(0,1)$ part $\tilde{D}^{(0,1)} := \pi^* D^{(0,1)} = \bar{\partial} + (\pi^* A)^{(0,1)}$ w.r.t. the complex structure on $\mathcal{Z}$. Its components along antiholomorphic vector fields on $\mathcal{U}_+$,

$$\bar{V}_{\bar{1}} = \partial_{\bar{y}} - \lambda\partial_z , \qquad \bar{V}_{\bar{2}} = \partial_{\bar{z}} + \lambda\partial_y , \qquad \bar{V}_{\bar{3}} = \partial_{\bar{\lambda}} , \tag{A.17}$$

have the form (see e.g. [58] and references therein)

$$\tilde{D}^{(0,1)}_{\bar{1}} = \bar{V}_{\bar{1}} + A_{\bar{y}} - \lambda A_z \ , \qquad \tilde{D}^{(0,1)}_{\bar{2}} = \bar{V}_{\bar{2}} + A_{\bar{z}} + \lambda A_y \ , \qquad \tilde{D}^{(0,1)}_{\bar{3}} = \bar{V}_{\bar{3}} \ , \tag{A.18}$$

which coincide with those in (A.7).

Holomorphic sections of π^*E. Let us consider local holomorphic sections s of the bundle $\pi^*E \to \mathcal{Z}$, which are defined by the equation $\tilde{D}^{(0,1)}s = 0$ or, in components,

$$\tilde{D}^{(0,1)}_{\bar{a}} s = 0 \qquad \text{for} \quad \bar{a} = 1,2,3\,. \tag{A.19}$$

The bundle $E' := \pi^*E$ is called holomorphic iff these equations are compatible in the sense that $(\tilde{D}^{(0,1)})^2 = 0 \ \Leftrightarrow \ [\tilde{D}^{(0,1)}_{\bar{a}}, \tilde{D}^{(0,1)}_{\bar{b}}] = 0$ for $\bar{a}, \bar{b}, \ldots = 1,2,3$. These compatibility equations coincide with the SDYM equations (A.5). So, if the gauge field is self-dual then there exist local solutions $s_\pm$ on $\mathcal{U}_\pm$ with $s_+ = s_-$ on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$. Note that one can always represent $s_\pm$ in the form $s_\pm = \psi_\pm \chi_\pm$, where $\psi_\pm$ are $n \times n$ matrix-valued nonsingular functions on $\mathcal{U}_\pm$ satisfying $\tilde{D}^{(0,1)}\psi_\pm = 0$, and $\chi_\pm \in \mathbb{C}^n$ are vector functions satisfying $\bar{\partial}\chi_\pm = 0$.

Holomorphic transition functions. The vector functions $\chi_\pm \in \mathbb{C}^n$ are holomorphic on $\mathcal{U}_\pm$, i.e. they are only functions of w_a and $\tilde{w}_a$, respectively. Furthermore, on $\mathcal{U}_+ \cap \mathcal{U}_-$ they are related via

$$\chi_+ = f_{+-}\, \chi_- \tag{A.20}$$

with

$$f_{+-} = \psi_+^{-1}\, \psi_- \tag{A.21}$$

which implies holomorphicity of f_{+-} on $\mathcal{U}_+ \cap \mathcal{U}_-$, i.e.

$$(\partial_{\bar{y}} - \lambda\partial_z) f_{+-} = 0\,, \qquad (\partial_{\bar{z}} + \lambda\partial_y) f_{+-} = 0 \qquad \text{and} \qquad \partial_{\bar{\lambda}} f_{+-} = 0\,. \tag{A.22}$$

One may restrict this function to the manifold $\mathcal{Z}_0 = \mathbb{R}^4 \times S^1 \subset \mathcal{U}_+ \cap \mathcal{U}_- \subset \mathcal{Z}$ on which it will be real-analytic. Any such function defines a holomorphic bundle E' over $\mathcal{Z}$. Namely, f_{+-} can be identified with a transition function in the holomorphic bundle π^*E over $\mathcal{Z}$, and the pair of functions $\psi_\pm$ in (A.21) defines a smooth trivialization of this bundle.

Reality conditions. The reality of the gauge fields is an important issue [45, 56]. The antihermiticity conditions $A_\mu^\dagger = -A_\mu$ for the components of the gauge potential can be satisfied by imposing the following 'reality' conditions for the matrices f_{+-} and $\psi_\pm$:

$$f^\dagger_{+-}(x, -\bar{\lambda}^{-1}) = f_{+-}(x,\lambda)\,, \tag{A.23}$$
$$\psi^\dagger_+(x, -\bar{\lambda}^{-1}) = \psi_-^{-1}(x,\lambda)\,. \tag{A.24}$$

It is easy to see that the gauge transformations

$$A_\mu \mapsto A_\mu^g = g^{-1} A_\mu\, g + g^{-1}\partial_\mu\, g \tag{A.25}$$

are induced (via (A.7)) by the transformations

$$\psi_+ \mapsto \psi_+^g = g^{-1}\psi_+ \qquad \text{and} \qquad \psi_- \mapsto \psi_-^g = g^{-1}\psi_-\ , \tag{A.26}$$

where $g{=}g(x)$ is an arbitrary $\mathrm{U}(n)$-valued function on $\mathbb{R}^4$. The transition function $f_{+-} = \psi_+^{-1}\psi_-$ is obviously invariant under these transformations. Hence, we have shown that one can associate a transition function f_{+-} in the holomorphic vector bundle π^*E over the twistor space $\mathcal{Z}$ to a gauge equivalence class $[A]$ of solutions to the SDYM equations on $\mathbb{R}^4$. In other words, we have described a map from complex vector bundles E over $\mathbb{R}^4$ with self-dual connections (modulo gauge transformations) into topologically trivial holomorphic vector bundles $E' = \pi^*E$ over the twistor space $\mathcal{Z} \cong \mathbb{R}^4 \times S^2$.

Splitting of transition functions. Consider now the converse situation. Let E' be a topologically trivial holomorphic bundle over $\mathcal{Z}$ with a real-analytic transition function f_{+-} on $\mathcal{Z}_0 = \mathbb{R}^4 \times S^1 \subset \mathcal{Z}$. Suppose that a restriction of E' to any projective line $\mathbb{C}P^1_x \hookrightarrow \mathcal{Z}$ (a section (A.14) of the bundle (A.13)) is holomorphically trivial. Then for each fixed $x \in \mathbb{R}^4$ one can find matrix-valued functions $\psi_\pm(x,\lambda)$ such that $f_{+-}(x,\lambda)$ as a function of $\lambda \in S^1 \subset \mathbb{C}P^1$ can be split as $f_{+-}(x,\lambda) = \psi_+^{-1}(x,\lambda)\psi_-(x,\lambda)$, where ψ_+ and ψ_- are boundary values of holomorphic functions on (subsets of) $U_+ \subset \mathbb{C}P^1$ and $U_- \subset \mathbb{C}P^1$, respectively. For a fixed point $x \in \mathbb{R}^4$, the task to split a matrix-valued real-analytic function $f_{+-}(x,\lambda) = f_{+-}(y{-}\lambda\bar z, z{+}\lambda\bar y, \lambda)$ on

$$S^1_x = \{\lambda \in \mathbb{C}P^1 \ : \ |\lambda| = 1,\ w_1 = y - \lambda\bar z,\ w_2 = z + \lambda\bar y\} \tag{A.27}$$

defines a parametric Riemann-Hilbert problem on $\mathbb{C}P^1_x \hookrightarrow \mathcal{Z}$. If its solution exists for given $x \in \mathbb{R}^4$ then one can prove that solutions exist also on an open neighbourhood U of x in $\mathbb{R}^4$ [23].

From the holomorphicity of f_{+-} it follows that $f_{+-}(x,\lambda) = f_{+-}(y{-}\lambda\bar z, z{+}\lambda\bar y, \lambda)$ satisfies (A.22). After splitting $f_{+-}(x,\lambda) = \psi_+^{-1}(x,\lambda)\psi_-(x,\lambda)$ we obtain

$$\psi_+(\partial_{\bar y} - \lambda\partial_z)\psi_+^{-1} = \psi_-(\partial_{\bar y} - \lambda\partial_z)\psi_-^{-1} \qquad \text{and} \qquad \psi_+(\partial_{\bar z} + \lambda\partial_y)\psi_+^{-1} = \psi_-(\partial_{\bar z} + \lambda\partial_y)\psi_-^{-1}\,. \tag{A.28}$$

We expand ψ_+ and ψ_- into power series in λ and λ^{-1}, respectively. Upon substituting into (A.28) one easily sees that both sides of (A.28) must be linear in λ, and one can introduce Lie-algebra valued fields A_μ by

$$A_{\bar y} - \lambda A_z := \psi_\pm(\partial_{\bar y} - \lambda\partial_z)\psi_\pm^{-1} \qquad \text{and} \qquad A_{\bar z} + \lambda A_y := \psi_\pm(\partial_{\bar z} + \lambda\partial_y)\psi_\pm^{-1}\,. \tag{A.29}$$

Hence, the gauge potential components may be calculated from

$$A_{\bar y} = \psi_+\partial_{\bar y}\psi_+^{-1}|_{\lambda=0} = -A_y^\dagger \qquad \text{and} \qquad A_{\bar z} = \psi_+\partial_{\bar z}\psi_+^{-1}|_{\lambda=0} = -A_z^\dagger \tag{A.30}$$

if $\psi_\pm$ satisfy the reality condition (A.24). By construction, the components A_μ of the gauge potential A defined above satisfy the SDYM equations. Note that the gauge potential A is inert under the transformations

$$\psi_+ \mapsto \psi_+\, h_+^{-1}\,, \qquad \psi_- \mapsto \psi_-\, h_-^{-1}\,, \qquad f_{+-} \mapsto h_+ f_{+-} h_-^{-1}\,, \tag{A.31}$$

where $h_+{=}h_+(w_1,w_2,\lambda)$ and $h_-{=}h_-(\tilde w_1,\tilde w_2,\tilde\lambda)$ are arbitrary matrix-valued holomorphic functions on $\mathcal{U}_+$ and $\mathcal{U}_-$, respectively. Two transition functions are said to be equivalent if they are related by the transformation (A.31). Thus, it defines an equivalence class $[E']$ of holomorphic bundles E'.

Noncommutative twistor correspondence. To sum up, we have described a one-to-one correspondence between pairs $(E,[D])$ and $([E'],\bar\partial)$, where E is a complex vector bundle over $\mathbb{R}^4$, $[D]$ are gauge equivalence classes of self-dual connections on E, and $[E']$ are equivalence classes of holomorphic vector bundles E' over the twistor space $\mathcal{Z}$ for $\mathbb{R}^4$ which are holomorphically trivial on each real projective line $\mathbb{C}P^1_x$ in $\mathcal{Z}$, $x\in U\subset\mathbb{R}^4$. This so-called *twistor correspondence* can be described by the following diagram,

$$\begin{tikzcd} E' \arrow[r] & \mathcal{Z} \arrow[r, "p"] \arrow[d, "\pi"'] & \mathbb{C}P^1 \\ E \arrow[r] & \mathbb{R}^4 & \end{tikzcd} \tag{A.32}$$

Note that instead of the complex vector bundle E and holomorphic vector bundle E' one can consider the space $\mathcal{E}=\mathcal{A}(\mathbb{R}^4)\otimes\mathbb{C}^n$ of (smooth or real-analytic) sections of E and the sheaf $\mathcal{E}'$ of holomorphic sections of E', respectively. Then the twistor correspondence can be reformulated as a correspondence between pairs $(\mathcal{E},[D])$ and $(\mathcal{E}',\bar\partial)$. To obtain a noncommutative deformation of the twistor correspondence one should simply introduce sheaves $\mathcal{E}_\theta$ and $\mathcal{E}'_\theta$ (parametrized by some constant matrix θ) which are deformations of $\mathcal{E}$ and $\mathcal{E}'$, respectively [59].

Deformed sheaves of functions. For a realization of the above-mentioned deformations, consider the tensor algebra $T(\mathbb{R}^4)$ over $\mathbb{R}^4$ and a two-sided ideal I in $T(\mathbb{R}^4)$ generated by elements of the form

$$\hat r^{\mu\nu}:=\hat x^\mu\hat x^\nu-\hat x^\nu\hat x^\mu-\mathrm{i}\theta^{\mu\nu}\,, \tag{A.33}$$

where $\theta=(\theta^{\mu\nu})$ is an antisymmetric constant matrix. Then, define the algebra $\mathcal{A}_\theta(\mathbb{R}^4)$ as an algebra over $\mathbb{C}$ generated by $\hat x^\mu$ with the relations

$$[\hat x^\mu,\hat x^\nu]=\mathrm{i}\theta^{\mu\nu}\,, \tag{A.34}$$

i.e. as the quotient algebra

$$\mathcal{A}_\theta(\mathbb{R}^4)=T(\mathbb{R}^4)/I \tag{A.35}$$

usually called the Weyl algebra. Deviating from the standard notation, we denote by $\mathbb{R}^4_\theta$ the real vector space spanned by $\hat x^1,\hat x^2,\hat x^3,\hat x^4$. Clearly, $\mathbb{R}^4_\theta\subset\mathcal{A}_\theta(\mathbb{R}^4)$. Analogously, we denote by $\mathbb{C}^2_\theta(\lambda)$ and $\mathbb{C}^2_\theta(\tilde\lambda)$ the complex vector spaces spanned by $\hat w_1=\hat y-\lambda\hat{\bar z}$ and $\hat w_2=\hat z+\lambda\hat{\bar y}$ with $\lambda\in U_+\subset\mathbb{C}P^1$, and by $\hat{\tilde w}_1=\tilde\lambda\hat y-\hat{\bar z}$ and $\hat{\tilde w}_2=\tilde\lambda\hat z+\hat{\bar y}$ with $\tilde\lambda\in U_-\subset\mathbb{C}P^1$, respectively. As in the commutative case, $\hat w_{1,2}$ and $\hat{\tilde w}_{1,2}$ are related on $U_+\cap U_-$ by $\hat w_{1,2}=\lambda\hat{\tilde w}_{1,2}$, i.e. they are operator-valued sections over U_+ and U_- of the holomorphic bundle $\mathcal{O}(1)\otimes\mathbb{C}^2$. Their commutation relations follow from (A.34). They generate algebras $\mathcal{O}_\theta(\mathbb{C}^2(\lambda))\supset\mathbb{C}^2_\theta(\lambda)$ and $\mathcal{O}_\theta(\mathbb{C}^2(\tilde\lambda))\supset\mathbb{C}^2_\theta(\tilde\lambda)$ of holomorphic functions of $\hat w_1,\hat w_2$ and $\hat{\tilde w}_1,\hat{\tilde w}_2$, respectively. These algebras are subalgebras in the algebras $\mathcal{O}_\theta(\mathcal{U}_\pm)$ of local sections of the holomorphic sheaf $\mathcal{O}_\theta$ over $\mathcal{U}_\pm\subset\mathcal{Z}$ generated by $\hat w_a$ and $\hat{\tilde w}_a$. Note that we deform the spaces $\mathcal{E}'(\mathcal{U}_\pm)$ of local holomorphic sections of E' over $\mathcal{U}_+$ and $\mathcal{U}_-$ instead of considering global smooth (or real-analytic) sections and a projector encoding transition functions f_{+-} in E' via the Serre-Swan theorem [60]. The reason is that the Riemann sphere $\mathbb{C}P^1$ in (A.32) is not deformed,

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and we consider operator-valued functions on $U_\pm \subset \mathbb{C}P^1$ glued on the overlap $U_+ \cap U_-$ by operator transition functions $\hat{f}_{+-}(\hat{x},\lambda)$ when discussing Riemann-Hilbert problems and Birkhoff factorizations.

Deformed twistor space. One can introduce the deformed twistor space as $\mathcal{Z}'_\theta := \mathcal{A}_\theta(\mathbb{R}^4) \times \mathbb{C}P^1$ with a double fibration

$$\begin{array}{ccc} \mathcal{A}_\theta(\mathbb{R}^4) \times \mathbb{C}P^1 & \xrightarrow{p'} & \mathbb{C}P^1 \\ {\scriptstyle \pi'}\big\downarrow & & \\ \mathcal{A}_\theta(\mathbb{R}^4) & & \end{array} \tag{A.36}$$

where the fibres of the holomorphic projection p' are the algebras $\mathcal{O}_\theta(\mathbb{C}^2(\lambda)) = p'^{-1}(\lambda)$ for $\lambda \in \mathbb{C}P^1$. Instead of (A.36) one may also consider the space $\mathcal{Z}_\theta := \mathbb{R}^4_\theta \times \mathbb{C}P^1$ with two projections,

$$\begin{array}{ccc} \mathbb{R}^4_\theta \times \mathbb{C}P^1 & \xrightarrow{p} & \mathbb{C}P^1 \\ {\scriptstyle \pi}\big\downarrow & & \\ \mathbb{R}^4_\theta & & \end{array} \tag{A.37}$$

for a tighter connection with the commutative case. In (A.37) the fibres over $\mathbb{C}P^1$ are the vector spaces $\mathbb{C}^2_\theta(\lambda') = p^{-1}(\lambda')$ for $\lambda' \in \mathbb{C}P^1$.

Operator Riemann-Hilbert problems. We will not discuss the noncommutative twistor correspondence in detail since it mainly repeats the commutative one with $x^\mu \mapsto \hat{x}^\mu$. In basic terms, there is a correspondence between gauge equivalence classes of operators $\hat{A}_\mu(\hat{x})$ solving the noncommutative SDYM equations and holomorphic equivalence classes of (operator-valued) transition functions $\hat{f}_{+-}(\hat{x},\lambda) = \hat{f}_{+-}(\hat{y}-\lambda\hat{\bar{z}}, \hat{z}+\lambda\hat{\bar{y}}, \lambda)$ in the complex vector bundle over $\mathbb{C}P^1$ with fibres $\mathcal{O}_\theta(\mathbb{C}^2(\lambda)) \otimes \mathbb{C}^n$ at $\lambda \in \mathbb{C}P^1$, such that $\hat{f}_{+-}(\hat{x},\lambda) = \hat{\psi}_+^{-1}(\hat{x},\lambda)\hat{\psi}_-(\hat{x},\lambda)$ on $S^1 \subset \mathbb{C}P^1$ and one has

$$\hat{A}_{\bar{y}} - \lambda\hat{A}_z = \hat{\psi}_\pm(\partial_{\bar{y}} - \lambda\partial_z)\hat{\psi}_\pm^{-1} \qquad \text{and} \qquad \hat{A}_{\bar{z}} + \lambda\hat{A}_y = \hat{\psi}_\pm(\partial_{\bar{z}} + \lambda\partial_y)\hat{\psi}_\pm^{-1}\,. \tag{A.38}$$

Here, $\hat{\psi}_+(\hat{x},\lambda)$ and $\hat{\psi}_-(\hat{x},\lambda)$ are boundary values of a holomorphic function (in λ) on (subsets of) $U_+ \subset \mathbb{C}P^1$ and $U_- \subset \mathbb{C}P^1$, respectively. They solve the operator Riemann-Hilbert problem of splitting $\hat{f}_{+-}(\hat{x},\lambda)$ on $S^1 = \{\lambda \in \mathbb{C}P^1 : |\lambda| = 1\}$. From (A.38) one can either find $\hat{\psi}_\pm(\hat{x},\lambda)$ and therefore $\hat{f}_{+-}(\hat{x},\lambda)$ if the $\hat{A}_\mu(\hat{x})$ are given or, conversely, find the $\hat{A}_\mu(\hat{x})$ if $\hat{f}_{+-}(\hat{x},\lambda)$ and its Birkhoff decomposition are known.

B. Monopoles in the twistor approach

Translational invariance. As was briefly mentioned in appendix A, the Bogomolny equations in $\mathbb{R}^3$ can be obtained from the self-dual Yang-Mills (SDYM) equations in $\mathbb{R}^4$ by imposing on the gauge potential $\{A_\mu, \mu = 1,\dots,4\}$ a condition of invariance w.r.t. translation along x^4-axis and by putting $\phi := A_4$. Here we are going to consider the twistor description of monopoles which follows from the description of self-dual gauge fields discussed in appendix A.

The translation generated by the vector field $T = \partial/\partial x^4$ is an isometry of $\mathbb{R}^4$ and yields a free twistor space action of the additive group $\mathbb{R}$ which is the real part of the holomorphic action of the complex number $\mathbb{C}$. This means that smooth T-invariant functions on $\mathcal{Z}$ can be considered as 'free' functions on the manifold

$$Y = \mathbb{R}^3 \times S^2 \,. \tag{B.1}$$

At the same time, for holomorphic functions f on $\mathcal{Z}$ we have

$$Tf = \partial_4 f(w_a) = \partial_4 w_a \frac{\partial}{\partial w_a} f = -\mathrm{i}(\partial_{w_2} + \lambda \partial_{w_1}) f(w_a) =: T' f \,, \tag{B.2}$$

where

$$T' = -\mathrm{i}(\partial_{w_2} + \lambda \partial_{w_1}) = -\mathrm{i}(\partial_{\tilde{w}_1} + \tilde{\lambda} \partial_{\tilde{w}_2}) \,. \tag{B.3}$$

In other words, $T = T' + \overline{T}'$, where the bar denotes complex conjugation. The first expression in (B.3) is valid on $\mathcal{U}_+$ and the second one on $\mathcal{U}_-$ (see appendix A for the definition of $w_a, \tilde{w}_a, \mathcal{U}_\pm$ and $\mathcal{Z}$). So, T'-invariant holomorphic functions on $\mathcal{Z}$ can be considered as 'free' holomorphic functions on a reduced twistor space $\mathcal{T}$ (called minitwistor space [22, 61]) obtained as the quotient space of $\mathcal{Z}$ by the action of the complex abelian symmetry group generated by T'.

Minitwistor space $\mathcal{T}$. It is not difficult to see that $\mathcal{T}$ is covered by two coordinate patches V_+ and V_- with coordinates $(\eta = w_1 - \lambda w_2, \lambda)$ and $(\tilde{\eta} = \tilde{w}_2 - \tilde{\lambda}\tilde{w}_1, \tilde{\lambda})$, respectively. These coordinates are constant along the T'-orbits in $\mathcal{Z}$, and on the overlap $V_+ \cap V_-$ we have

$$\eta = \lambda^2 \tilde{\eta} \qquad \text{and} \qquad \lambda = \tilde{\lambda}^{-1} \tag{B.4}$$

which coincides with the transformation of coordinates on the total space of the holomorphic line bundle $\mathcal{O}(2) \to \mathbb{C}P^1$ with $c_1(\mathcal{O}(2)) = 2$. In (B.4), λ and $\tilde{\lambda}$ are coordinates on $\mathbb{C}P^1$ (see appendix A), while η and $\tilde{\eta}$ are coordinates in the fibres over U_+ and U_-, respectively, where $U_+ \cup U_- = \mathbb{C}P^1$. So, as the minitwistor space we obtain the total space $\mathcal{T}$ of the bundle

$$p: \quad \mathcal{T} = \mathcal{O}(2) \cong T\mathbb{C}P^1 \;\to\; \mathbb{C}P^1 \,, \tag{B.5}$$

where $T\mathbb{C}P^1$ denotes the holomorphic tangent bundle of $\mathbb{C}P^1$. The space $\mathcal{T}$ is part of the double fibration

$$\begin{tikzcd} Y = \mathbb{R}^3 \times S^2 \arrow[r, "q"] \arrow[d, "\pi"'] & \mathcal{T} \\ \mathbb{R}^3 & \end{tikzcd} \tag{B.6}$$

where π projects onto $\mathbb{R}^3$ and $q(x,\lambda) = \{\eta = y - 2\lambda x^3 - \lambda^2 \bar{y}, \lambda\}$.

Real structure on $\mathcal{T}$. The real structure τ on $\mathcal{Z}$ (see appendix A) induces a real structure (antiholomorphic involution, $\tau^2 = 1$) on $\mathcal{T}$ acting on (local) coordinates as follows,

$$\tau(\eta, \lambda) = \left(-\frac{\bar{\eta}}{\bar{\lambda}^2}, -\frac{1}{\bar{\lambda}} \right) . \tag{B.7}$$

From this definition it is obvious that τ has no fixed points on $\mathcal{T}$ but does leave invariant the projective lines

$$\mathbb{C}P^1_x = \begin{cases} \lambda \in U_+ \,, & \eta = y - 2\lambda x^3 - \lambda^2 \bar{y} \\ \tilde{\lambda} \in U_- \,, & \tilde{\eta} = \tilde{\lambda}^2 y - 2\tilde{\lambda} x^3 - \bar{y} \end{cases} \tag{B.8}$$

which are real holomorphic sections of the bundle (B.5) parametrized by $x \in \mathbb{R}^3$. Note that the diagram (B.6) describes a one-to-one correspondence between points $x \in \mathbb{R}^3$ and projective lines (B.8) in $\mathcal{T}$. Conversely, points of $\mathcal{T}$ correspond to oriented lines in $\mathbb{R}^3$ [22, 61].

Minitwistor correspondence. Recall again that the Bogomolny equations (2.9) are a reduction to $\mathbb{R}^3$ of the SDYM equations on $\mathbb{R}^4$ which arise as compatibility conditions of the linear system (A.7) from the appendix A. One can discard the ignorable coordinate x^4 in the gauge potential and obtain a reduced linear system for which the compatibility conditions are equivalent to the Bogomolny equations. Therefore, a correspondence between solutions (A_a, ϕ) of the Bogomolny equations and holomorphic vector bundles E' over the minitwistor space $\mathcal{T}$ directly follows from the twistor correspondence described in appendix A. Simply, $(A_a, \phi{=}A_4)$ and f_{+-} should not depend on x^4, but x^4 dependence is allowed for $\psi_\pm(x,\lambda)$ especially if one abandons the reality condition (A.24) for them. Hence, there is a one-to-one correspondence between pairs $(E, [A_a, \phi])$ and $([E'], \bar{\partial})$, where E is a complex vector bundle over $\mathbb{R}^3$, $[A_a, \phi]$ are gauge equivalence classes of solutions to the Bogomolny equations, and $[E']$ are equivalence classes of holomorphic vector bundles over the minitwistor space $\mathcal{T}$ which are holomorphically trivial on each real projective line $\mathbb{C}P^1_x \hookrightarrow \mathcal{T}$, as defined by (B.8).

Riemann-Hilbert problems. Recall that any holomorphic vector bundle E' over $\mathcal{T}$ is defined by a transition function $f_{+-}(\eta,\lambda)$ on $V_+ \cap V_-$ whose restriction $f_{+-}(y{-}2\lambda x^3{-}\lambda^2\bar{y}, \lambda)$ to $\mathbb{C}P^1_x \hookrightarrow \mathcal{T}$ defines parametric Riemann-Hilbert problems. Therefore, the above-mentioned correspondence implies a correspondence between solutions (A_a, ϕ) of the Bogomolny equations and solutions $\psi_\pm(x,\lambda)$ of the Riemann-Hilbert problems on $\mathbb{C}P^1_x \hookrightarrow \mathcal{T}$, where $x \in U \subset \mathbb{R}^3$. Here, $\psi_\pm(x,\lambda)$ are the result of a Birkhoff decomposition $f_{+-}(x,\lambda) = \psi_+^{-1}(x,\lambda)\Lambda(x,\lambda)\psi_-(x,\lambda)$ with $\Lambda = 1$.

Note that $f_{+-}(\eta,\lambda) \equiv f_{+-}(w_1{-}\lambda w_2, \lambda)$ is annihilated by the vector fields (A.17). Therefore, from the (generalized) Liouville theorem it follows that

$$\begin{aligned} \psi_+(\partial_{\bar{y}} - \lambda\partial_z)\psi_+^{-1} &= \psi_-(\partial_{\bar{y}} - \lambda\partial_z)\psi_-^{-1} = \frac{1}{2}(A_1 + \mathrm{i}A_2) - \frac{\lambda}{2}(A_3 + \mathrm{i}\phi)\,, \\ \psi_+(\partial_{\bar{z}} + \lambda\partial_y)\psi_+^{-1} &= \psi_-(\partial_{\bar{z}} + \lambda\partial_y)\psi_-^{-1} = \frac{1}{2}(A_3 - \mathrm{i}\phi) + \frac{\lambda}{2}(A_1 - \mathrm{i}A_2)\,, \end{aligned} \tag{B.9}$$

where the fields (A_a, ϕ) do not depend on λ. Here we keep ∂_4 since, as mentioned above, $\psi_\pm(x,\lambda)$ may depend on x^4 even if f_{+-} and (A_a,ϕ) do not. So, if we know f_{+-} and its Birkhoff decomposition $f_{+-}(x,\lambda) = \psi_+^{-1}(x,\lambda)\psi_-(x,\lambda)$, then (A_a,ϕ) can be calculated from

$$\frac{1}{2}(A_1 + \mathrm{i}A_2) = \psi_+(x,\lambda)\partial_{\bar{y}}\psi_+^{-1}(x,\lambda)|_{\lambda=0}\,, \quad \frac{1}{2}(A_3 - \mathrm{i}\phi) = \psi_+(x,\lambda)\partial_{\bar{z}}\psi_+^{-1}(x,\lambda)|_{\lambda=0}\,,$$

$$\frac{1}{2}(A_1 - \mathrm{i}A_2) = \psi_-(x,\lambda)\partial_y\psi_-^{-1}(x,\lambda)|_{\lambda=\infty}\,, \quad \frac{1}{2}(A_3 + \mathrm{i}\phi) = \psi_-(x,\lambda)\partial_z\psi_-^{-1}(x,\lambda)|_{\lambda=\infty}\,. \tag{B.10}$$

By construction, the fields (B.10) satisfy the Bogomolny equations. Furthermore, (A_a, ϕ) are real (antihermitean) if $\psi_\pm(x,\lambda)$ satisfy the reality conditions (A.24).

Deformed sheaves of functions. For the noncommutative deformation of the minitwistor correspondence we consider the algebra $\mathcal{A}_\theta(\mathbb{R}^3)$ introduced in section 2.3,

$$\mathcal{A}_\theta(\mathbb{R}^3) = T(\mathbb{R}^3)/\mathcal{I}\,, \tag{B.11}$$

where $T(\mathbb{R}^3)$ is the tensor algebra of $\mathbb{R}^3$, and an ideal $\mathcal{I}$ is generated by elements of the form

$$\hat{x}^1\hat{x}^3 - \hat{x}^3\hat{x}^1\,, \qquad \hat{x}^2\hat{x}^3 - \hat{x}^3\hat{x}^2 \qquad \text{and} \qquad \hat{x}^1\hat{x}^2 - \hat{x}^2\hat{x}^1 - \mathrm{i}\theta \qquad \text{with} \quad \theta \geq 0\,. \tag{B.12}$$

Following the logic of appendix A, we denote by $\mathbb{R}^3_\theta$ the real vector space spanned on $\hat{x}^1, \hat{x}^2$ and $\hat{x}^3$. Obviously, $\mathbb{R}^3_\theta \subset \mathcal{A}_\theta(\mathbb{R}^3)$, and $\hat{x}^3$ is the operator of multiplication with x^3, which commutes with $\hat{x}^1$ and $\hat{x}^2$. We also denote by $\mathbb{C}_\theta(\lambda')$ a one-parameter space of operators

$$\hat{\eta}(\lambda) = \hat{y} - 2\lambda x^3 - \lambda^2\hat{\bar{y}} \quad \text{on} \quad U_+ \qquad \text{and} \qquad \hat{\tilde{\eta}}(\tilde{\lambda}) = \tilde{\lambda}^2\hat{y} - 2\tilde{\lambda}x^3 - \hat{\bar{y}} \quad \text{on} \quad U_-\,. \tag{B.13}$$

On the intersection $U_+ \cap U_-$ these operators are related by $\hat{\eta} = \lambda^2\hat{\tilde{\eta}}$ (cf. (B.4)) with $\lambda = \tilde{\lambda}^{-1}$, i.e. they are operator-valued sections of the bundle

$$\mathcal{T}_\theta \;\to\; \mathbb{C}P^1 \tag{B.14}$$

with $\mathbb{C}_\theta(\lambda')$ as fibres over $\lambda' \in \mathbb{C}P^1$. The spaces $\mathbb{C}_\theta(\lambda')$ generate a one parameter family of algebras $\mathcal{O}_\theta(\mathbb{C}(\lambda')) \supset \mathbb{C}_\theta(\lambda')$ of holomorphic operator-valued functions on the fibres of (B.14).

Deformed minitwistor space. Recall that the real holomorphic sections (B.8) of the bundle (B.5) are parametrized by the coordinates x^a on $\mathbb{R}^3$. Substituting operators $\hat{x}^a$ instead of coordinates x^a, one obtains sections (B.13) of the bundle (B.14) which may be considered as a noncommutative minitwistor space. Thus, the deformation of the minitwistor space $\mathcal{T}$ simply means the usage of operators $\hat{\eta}$ and $\hat{\tilde{\eta}}$ instead of coordinates η and $\tilde{\eta}$ on fibres of the bundle (B.14). However, this substitution does not change the mutual commutativity of the 'coordinates' $\hat{\eta}$, λ and $\hat{\tilde{\eta}}$, $\tilde{\lambda}$ on $\mathcal{T}_\theta$. Note that instead of $\mathcal{T}_\theta$ one may also consider for the deformed minitwistor space the bundle

$$\mathcal{T}'_\theta \;\to\; \mathbb{C}P^1 \tag{B.15}$$

with algebras $\mathcal{O}_\theta(\mathbb{C}(\lambda')) \supset \mathbb{C}_\theta(\lambda')$ as fibres at $\lambda' \in \mathbb{C}P^1$ (cf. discussion in appendix A). In other words, we deform only the fibres in our vector bundles over $\mathbb{C}P^1$ since the base manifold $\mathbb{C}P^1$ is unchanged. Hence, as a generalization of the diagram (B.6) one may use the double fibrations

$$\begin{array}{ccc} \mathcal{A}_\theta(\mathbb{R}^3)\times\mathbb{C}P^1 & \longrightarrow & \mathcal{T}'_\theta \\ \downarrow & & \\ \mathcal{A}_\theta(\mathbb{R}^3) & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{R}^3_\theta\times\mathbb{C}P^1 & \longrightarrow & \mathcal{T}_\theta \\ \downarrow & & \\ \mathbb{R}^3_\theta & & \end{array} \tag{B.16}$$

for a description of the noncommutative minitwistor correspondence.

Operator Riemann-Hilbert problems. Finally, a correspondence between solutions of the noncommutative Bogomolny equations and solutions of the operator Riemann-Hilbert problems is described by formulae generalizing (B.9),

$$\begin{aligned}\hat{\psi}_+(\hat{x},\lambda)(\partial_{\bar{y}}-\lambda\partial_z)\hat{\psi}_+^{-1}(\hat{x},\lambda) &= \hat{\psi}_-(\hat{x},\lambda)(\partial_{\bar{y}}-\lambda\partial_z)\hat{\psi}_-^{-1}(\hat{x},\lambda) = \frac{1}{2}(\hat{A}_1+\mathrm{i}\hat{A}_2)-\frac{\lambda}{2}(\hat{A}_3+\mathrm{i}\hat{\phi})\,, \\ \hat{\psi}_+(\hat{x},\lambda)(\partial_{\bar{z}}+\lambda\partial_y)\hat{\psi}_+^{-1}(\hat{x},\lambda) &= \hat{\psi}_-(\hat{x},\lambda)(\partial_{\bar{z}}+\lambda\partial_y)\hat{\psi}_-^{-1}(\hat{x},\lambda) = \frac{1}{2}(\hat{A}_3-\mathrm{i}\hat{\phi})+\frac{\lambda}{2}(\hat{A}_1-\mathrm{i}\hat{A}_2)\,.\end{aligned} \tag{B.17}$$

The derivation of these lines literally follows that of (B.9). Just note that now $\hat{f}_{+-}(\hat{y}-2\lambda x^3-\lambda^2\hat{\bar{y}},\lambda)$ is an operator transition function in a holomorphic vector bundle over $\mathbb{C}P^1$ with fibres $\mathcal{O}_\theta(\mathbb{C}(\lambda'))\otimes\mathbb{C}^n$ at $\lambda'\in\mathbb{C}P^1$. So, for a given $\hat{f}_{+-}(\hat{x},\lambda)=\hat{\psi}_+^{-1}(\hat{x},\lambda)\hat{\psi}_-(\hat{x},\lambda)$ one can calculate $(\hat{A}_a,\hat{\phi})$ from

$$\begin{aligned}\frac{1}{2}(\hat{A}_1+\mathrm{i}\hat{A}_2) &= \hat{\psi}_+(\hat{x},\lambda)\partial_{\bar{y}}\hat{\psi}_+^{-1}(\hat{x},\lambda)|_{\lambda=0}\,, & \frac{1}{2}(\hat{A}_3-\mathrm{i}\hat{\phi}) &= \hat{\psi}_+(\hat{x},\lambda)\partial_{\bar{z}}\hat{\psi}_+^{-1}(\hat{x},\lambda)|_{\lambda=0}\,, \\ \frac{1}{2}(\hat{A}_1-\mathrm{i}\hat{A}_2) &= \hat{\psi}_-(\hat{x},\tilde{\lambda})\partial_y\hat{\psi}_-^{-1}(\hat{x},\tilde{\lambda})|_{\tilde{\lambda}=0}\,, & \frac{1}{2}(\hat{A}_3+\mathrm{i}\hat{\phi}) &= \hat{\psi}_-(\hat{x},\tilde{\lambda})\partial_z\hat{\psi}_-^{-1}(\hat{x},\tilde{\lambda})|_{\tilde{\lambda}=0}\,.\end{aligned} \tag{B.18}$$

By construction, these fields satisfy the noncommutative Bogomolny equations.

C. General Wu-Yang multi-monopoles

In section 5.2 we have described the Wu-Yang multi-monopole solution generated by the transition function (5.11) with ρ given by (5.28). This transition function belongs to the so-called Atiyah-Ward $\mathcal{A}_1$ ansatz. More general Wu-Yang SU(2) multi-monopoles can be obtained from the Atiyah-Ward $\mathcal{A}_n$ ansatz with a transition function of the form

$$f_{+-}^{(n)WY} = \begin{pmatrix} \rho_n & \lambda^{-n} \\ -\lambda^n & 0 \end{pmatrix}, \tag{C.1}$$

where ρ_n coincides with the function $f_{+-}^{(n)D}=(f_{-+}^{(n)D})^{-1}$ (see (4.19)) generating the Dirac multi-monopole,

$$\rho_n = \prod_{k=1}^{n}\frac{\lambda}{\eta-h_k} = f_{+-}^{(n)D}\,. \tag{C.2}$$

Note that the splitting $f_{+-}^{(n)WY}=(\psi_+^{(n)})^{-1}\psi_-^{(n)}$ of the $\mathcal{A}_n$ matrix (C.1) is a more complicated task than for the $\mathcal{A}_1$ case considered in section 5, and we will not consider it here.

Via splitting of the matrix (C.1) one can obtain the generic Wu-Yang SU(2) multi-monopole configuration which is precisely equivalent to the Dirac multi-monopole (4.23). This statement can easily be proved via the twistor argument. Namely, let us consider two rank 2 holomorphic vector bundles over the minitwistor space $\mathcal{T}=T\mathbb{C}P^1$ with holomorphic transition functions f_{+-} and $\tilde{f}_{+-}$ (with operator entries in the noncommutative case) defined on an open subset of $V_+\cap V_-\subset\mathcal{T}$. These bundles are called equivalent if there exist matrix-valued holomorphic functions h_+ and h_- on V_+ and V_-, respectively, such that

$$h_+\,f_{+-}\,h_- = \tilde{f}_{+-}\,. \tag{C.3}$$

These transition functions define the same configuration since the splitting of $\tilde{f}_{+-}$ yields $\tilde{\psi}_+ = \psi_+ h_+^{-1}$ and $\tilde{\psi}_- = \psi_- h_-$, and the factors $h_\pm$ disappear from the equations due to holomorphicity.

After these remarks, we explicitly consider the transition function (C.1) and take

$$h_+ = \begin{pmatrix} \prod_{k=1}^{n} (\eta - h_k) & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda^n \rho_n & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$h_- = \begin{pmatrix} -\prod_{k=1}^{n} \big(\tilde{\eta} - h_k(\tilde{\lambda})\big) & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\lambda^{-2n} \prod_{k=1}^{n} \big(\eta - h_k(\lambda)\big) & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\lambda^{-n} \rho_n^{-1} & 1 \\ 1 & 0 \end{pmatrix}, \tag{C.4}$$

which are holomorphic on V_+ and V_-, respectively. Then, we have

$$\begin{aligned} h_+ \, f_{+-}^{(n)WY} h_- &= \begin{pmatrix} \lambda^n \rho_n^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_n & \lambda^{-n} \\ -\lambda^n & 0 \end{pmatrix} \begin{pmatrix} -\lambda^{-n} \rho_n^{-1} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \rho_n^{-1} & 0 \\ 0 & \rho_n \end{pmatrix} = \begin{pmatrix} (f_{+-}^{(n)D})^{-1} & 0 \\ 0 & f_{+-}^{(n)D} \end{pmatrix}, \end{aligned} \tag{C.5}$$

where $f_{+-}^{(n)D}$ defines the Dirac multi-monopole, and the diagonal matrix (C.5) describes the line bundle L (the U(1) gauge group) embedded into the rank 2 vector bundle (the SU(2) gauge group) as $L^{-1} \oplus L$. This proves the equivalence. The same formulae can be written in the noncommutative case since all calculations above are purely algebraic.

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